On the McCool group M_3 and its associated Lie algebra

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Abstract

We prove that the Lie algebra of the McCool group M_3 is torsion free. As a result we are able to give a presentation for the Lie algebra of M_3. Furthermore, M_3 is a Magnus group.

1 Introduction

Throughout this paper, by "Lie algebra", we mean Lie algebra over the ring of integers $\mathbb{Z}$. Let G be a group. We denote by (a,b) the commutator $(a,b) = a^{-1}b^{-1}ab$. For a positive integer c, let $\gamma_c(G)$ be the c-th term of the lower central series of G. The (restricted) direct sum of the quotients $\gamma_c(G)/\gamma_{c+1}(G)$ is the *associated graded Lie algebra* of G, $\text{Ł}(G) = \bigoplus_{c\geq 1}\gamma_c(G)/\gamma_{c+1}(G)$. The Lie bracket multiplication in $\text{Ł}(G)$ is defined as $[a\gamma_{c+1}(G), b\gamma_{d+1}(G)] = (a,b)\gamma_{c+d+1}(G)$, with $a \in \gamma_c(G)$, $b \in \gamma_d(G)$ and $(a,b) \in \gamma_{c+d}(G)$ and extends the multiplication linearly.

For a group G, we write $\text{IA}(G)$ for the kernel of the natural group homomorphism from $\text{Aut}(G)$ into $\text{Aut}(G/G')$ with $G' = \gamma_2(G)$. For a positive integer $c \geq 2$, the natural group epimorphism from G onto $G/\gamma_c(G)$ induces a group homomorphism π_c from the automorphism group $\text{Aut}(G)$ into the automorphism group $\text{Aut}(G/\gamma_c(G))$. Write $\text{I}_c\text{A}(G) = \text{Ker}\pi_c$. Note that $\text{I}_2\text{A}(G) = \text{IA}(G)$. It is proved by Andreadakis [1, Theorem 1.2], that if G is residually nilpotent (that is, $\bigcap_{c\geq 1}\gamma_c(G) = \{1\}$) then $\bigcap_{c\geq 2}\text{I}_c\text{A}(G) = \{1\}$. For a positive integer n, with $n \geq 2$, we write F_n for a free group of rank n with a free generating set $\{x_1, \ldots, x_n\}$. It was shown by Magnus [16], using work of Nielsen [18], that $\text{IA}(F_n)$ has a finite generating set $\{\chi_{ij}, \chi_{ijk} : 1 \leq i,j,k \leq n; i \neq j,k; j < k\}$, where χ_{ij} maps $x_i \mapsto x_i(x_i, x_j)$ and χ_{ijk} maps $x_i \mapsto x_i(x_j^{-1}, x_k^{-1})$, with both χ_{ij} and χ_{ijk} fixing the remaining basis elements. Let M_n be the subgroup of $\text{IA}(F_n)$ generated by the subset $S = \{\chi_{ij} : 1 \leq i,j \leq n;\ i \neq j\}$. Then M_n is called the *McCool group* or the *basis conjugating automorphisms group*. It is easily verified that the following relations are satisfied by the elements of S, provided that, in each case, the subscripts i,j,k,q occurring are distinct:

$$\begin{aligned}(\chi_{ij}, \chi_{kj}) &= 1\\ (\chi_{ij}, \chi_{kq}) &= 1 \qquad (1)\\ (\chi_{ij}\chi_{kj}, \chi_{ik}) &= 1.\end{aligned}$$

It has been proved in [17] that M_n has a presentation $\langle S \mid Z\rangle$, where Z is the set of all possible relations of the above forms. Since $\gamma_c(M_n) \subseteq \gamma_c(\text{IA}(F_n)) \subseteq \text{I}_{c+1}\text{A}(F_n)$ for all $c \geq 1$, and since F_n is residually nilpotent, we have $\bigcap_{c\geq 1}\gamma_c(M_n) = \{1\}$ and so, M_n is residually nilpotent.

The study of M_n is strongly connected to that of B_n, the braid group. It is well known that M_n and B_n, as subgroups of $\text{Aut}(F_n)$ intersect at P_n, the pure braid group. The graded

algebra $\mathrm{gr}(P_n)$ has been studied extensively and Kohno [12] and Falk and Randell [10] provide an important description of these Lie algebras. Using similar techniques, Cohen et. al. [9] show that the graded Lie algebra of the upper triangular McCool group, $\mathrm{gr}(M_n^+)$, is additively isomorphic to the direct sum of free Lie subalgebras

$$\bigoplus L[\chi_{k1}, \dots, \chi_{k,k-1}]$$

with $[\chi_{kj}, \chi_{st}] = 0$ if $\{i,j\} \cup \{s,t\} = \emptyset$, $[\chi_{kj}, \chi_{sj}] = 0$ if $\{s,k\} \cap \{j\} = \emptyset$ and $[\chi_{ik}, \chi_{ij} + \chi_{kj}] = 0$ for $j < k < i$.

The description of the Lie algebra $\mathrm{gr}(M_n)$, with $n \geq 2$, which is stated as a problem in [3], is a non trivial problem. Since $M_2 = \mathrm{IA}(F_2) \cong F_2$, it is well known that $\mathrm{gr}(M_2)$ is a free Lie algebra of rank 2. For $n \geq 3$, it seems that the known techniques for $\mathrm{gr}(P_n)$ are not able to prove analogous results. In the present paper, we concentrate on M_3 and we perform a thorough analysis on its Lie algebra $\mathrm{gr}(M_3)$, in order to understand its structure. For that we use the presentation of M_3 given by McCool in [17]. Our analysis implies the following.

Theorem 1 *Let $gr(M_3)$ be the graded Lie algebra of M_3. Then*

1. *$gr(M_3)$ is torsion-free $\mathbb{Z}$-module. In particular, $\mathrm{gr}(M_3) \cong L/J$ as Lie algebras, where L is a free Lie algebra of rank 6 and J is a free Lie algebra.*

2. *M_3 is residually nilpotent and each $\gamma_c(M_3)/\gamma_{c+1}(M_3)$ is torsion free. That is, M_3 is a Magnus group.*

In the next few lines, we briefly explain our approach to prove Theorem 1. We use Lazard elimination to decompose the free Lie algebra L on six generators $\{x_1, \dots, x_6\}$ into the free Lie algebras on the $\mathbb{Z}$-modules V_i generated by $\{x_{2i-1}, x_{2i}\}$ (with $i = 1, 2, 3$) and the free Lie algebra on some specific $\mathbb{Z}$-module W. As W is graded, $W = \bigoplus_{n \geq 2} W_n$ and we decompose each W_n according to the number of generators of V_i that appear in its generators. So we describe in detail the generating sets of W_n. This allow us to describe a generating set for the derived algebra L'. Next, we use a Lie algebra automorphism of L' to change the generating set of L' in such a way that the relations of $\L(M_3)$ induced by the presentation of M_3 become part of the new generating set of L'. Again, we decompose L' using the new generating set. This decomposition, although long and tedious, is necessary in order to help us understand J, the ideal of L generated by the relations of M_3 viewed in L. We describe the components of J studying their intersection with the components of L'. In Theorem 2, our key result, we show that the components of J are direct summands of the components of L. For this we need to study the homogeneous components of J by means of Lyndon polynomials, a filtration of tensor powers and symmetric powers. As a consequence, L/J is a torsion-free $\mathbb{Z}$-module. By a result of Witt (see [2, Theorem 2.4.2.5]), we have J is a free Lie algebra. Finally, we prove that L/J is isomorphic to the Lie algebra of M_3. We should point out that the above method does not work for $n > 3$, due to the complexity of the calculations.

2 Preliminary results

In this section we give some preliminary results that will help us decompose $\L(M_3)$.

2.1 Some notation

Given a free $\mathbb{Z}$-module A, we write $L(A)$ for the free Lie algebra on A, that is the free Lie algebra on $\mathcal{A}$ where $\mathcal{A}$ is an arbitrary $\mathbb{Z}$-basis of A. Thus we may write $L(A) = L(\mathcal{A})$. For a positive integer c, let $L^c(A)$ denote the c-th homogeneous component of $L(A)$. It is well-known that

$$L(A) = \bigoplus_{c \geq 1} L^c(A).$$

Throughout this paper, we write $L = L(\mathcal{X})$ for the free Lie algebra of rank 6 with a free generating set $\mathcal{X} = \{x_1, \ldots, x_6\}$. The elements of $\mathcal{X}$ are ordered as $x_1 < x_2 < \cdots < x_6$. For a positive integer c, we write $L^c = L^c(\mathcal{X})$. From now on, we write

$$\begin{array}{llll} y_1 = [x_2, x_1], & y_2 = [x_4, x_3], & y_3 = [x_6, x_5], & \\ y_4 = [x_3, x_1], & y_5 = [x_4, x_1], & y_6 = [x_3, x_2], & y_7 = [x_4, x_2], \\ y_8 = [x_5, x_1], & y_9 = [x_6, x_1], & y_{10} = [x_5, x_2], & y_{11} = [x_6, x_2], \\ y_{12} = [x_5, x_3], & y_{13} = [x_6, x_3], & y_{14} = [x_5, x_4], & y_{15} = [x_6, x_4]. \end{array} \qquad (D)$$

Let J be the ideal of L generated by the set

$$\mathcal{V} = \{y_1, y_2, y_3, y_6 + y_7, y_7 + y_5, y_9 + y_8, y_{10} + y_8, y_{12} + y_{13}, y_{15} + y_{13}\}.$$

Let F be a free group of rank 6 with a free generating set $\mathcal{Y}_F = \{a_1, \ldots, a_6\}$. We order the elements of $\mathcal{Y}_F$ as $a_1 < a_2 < \cdots < a_6$. It is well known that $\L(F)$ is a free Lie algebra of rank 6; freely generated by the set $\{a_i F' : i = 1, \ldots, 6\}$. The free Lie algebras L and $\L(F)$ are isomorphic to each other by a natural isomorphism χ subject to $\chi(x_i) = a_i F'$, $i = 1, \ldots, 6$. From now on, we identify L with $\L(F)$, and write $x_i = a_i F'$, $i = 1, \ldots, 6$, and for $c \geq 2$,

$$[x_{i_1}, \ldots, x_{i_c}] = (a_{i_1}, \ldots, a_{i_c})\gamma_{c+1}(F)$$

for all $i_1, \ldots, i_c \in \{1, \ldots, 6\}$. Furthermore, for each $c \geq 1$, $L^c = \gamma_c(F)/\gamma_{c+1}(F)$. Define

$$\begin{array}{ll} r_1 = (a_2, a_1), r_2 = (a_4, a_3), & r_3 = (a_6, a_5) \\ r_4 = (a_1 a_2, a_5), & r_5 = (a_3 a_4, a_6) \\ r_6 = (a_2 a_1, a_4), & r_7 = (a_4 a_3, a_2) \\ r_8 = (a_6 a_5, a_3), & r_9 = (a_5 a_6, a_1), \end{array}$$

and $\mathcal{R} = \{r_1, \ldots, r_9\}$. Under the above identification, we have

$$\begin{array}{lcl} y_1 & = & r_1\gamma_3(F), \\ y_2 & = & r_2\gamma_3(F), \\ y_3 & = & r_3\gamma_3(F), \\ y_6 + y_7 & = & r_6\gamma_3(F), \\ y_7 + y_5 & = & r_7\gamma_3(F), \\ y_9 + y_8 & = & r_9\gamma_3(F), \\ y_{10} + y_8 & = & r_4\gamma_3(F), \\ y_{12} + y_{13} & = & r_8\gamma_3(F), \\ y_{15} + y_{13} & = & r_5\gamma_3(F). \end{array}$$

Let $N = \mathcal{R}^F$ be the normal closure of $\mathcal{R}$ in F. Thus N is generated as a group by the set $\{r^g = g^{-1}rg : r \in \mathcal{R}, g \in F\}$. By using the presentation of M_3, we may show that $M_3 = F/N$. Since $r \in F' \setminus \gamma_3(F)$ for all $r \in \mathcal{R}$, we have $N \subseteq F'$ and so, $M_3/M_3' \cong F/F'$.

2.2 Lazard elimination

For $\mathbb{Z}$-submodules A and B of any Lie algebra over $\mathbb{Z}$, let $[A, B]$ be the $\mathbb{Z}$-submodule spanned by $[a, b]$ where $a \in A$ and $b \in B$. Furthermore, $B \wr A$ denotes the $\mathbb{Z}$-submodule defined by

$$B \wr A = B + [B, A] + [B, A, A] + \cdots.$$

We use the left-normed convention for Lie commutators. One of our main tools in this paper is Lazard elimination. The following result is a version of Lazard's "Elimination Theorem" (see [4, Chapter 2, Section 2.9, Proposition 10]). In the form written here it is a special case of ([5, Lemma 2.2]) or ([7, Lemma 2]).

Lemma 1 *Let U and V be free $\mathbb{Z}$-modules, and consider the free Lie algebra $L(U \oplus V)$. Then U and $V \wr U$ freely generate Lie subalgebras $L(U)$ and $L(V \wr U)$, and there is a $\mathbb{Z}$-module decomposition $L(U \oplus V) = L(U) \oplus L(V \wr U)$. Furthermore,*

$$V \wr U = V \oplus [V, U] \oplus [V, U, U] \oplus \cdots$$

and, for each $n \geq 0$, there is a $\mathbb{Z}$-module isomorphism

$$\zeta_n : [V, \underbrace{U, \ldots, U}_{n}] \longrightarrow V \otimes \underbrace{U \otimes \cdots \otimes U}_{n}$$

such that $\zeta_n([v, u_1, \ldots, u_n]) = v \otimes u_1 \otimes \cdots \otimes u_n$ for all $v \in V$ and $u_1, \ldots, v_n \in U$.

As a consequence of Lemma 1 we have the following result. This is a special case of [7, Proof of Lemma 3].

Corollary 1 *For free $\mathbb{Z}$-modules $V_1, \ldots, V_n$, with $n \geq 2$, we write $L(V_1 \oplus \cdots \oplus V_n)$ for the free Lie algebra on $V_1 \oplus \cdots \oplus V_n$. Then there is a $\mathbb{Z}$-module decomposition $L(V_1 \oplus \cdots \oplus V_n) = L(V_1) \oplus \cdots \oplus L(V_n) \oplus L(W)$, where $W = W_2 \oplus W_3 \oplus \cdots$ such that, for all $m \geq 2$, W_m is the direct sum of submodules $[V_{i_1}, V_{i_2}, V_{i_3}, \ldots, V_{i_m}]$ $(i_1 > i_2 \leq i_3 \leq \cdots \leq i_m)$. Each $[V_{i_1}, V_{i_2}, V_{i_3}, \ldots, V_{i_m}]$ is isomorphic to $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_m}$ as $\mathbb{Z}$-module. Furthermore, $L(W)$ is the ideal of $L(V_1 \oplus \cdots \oplus V_n)$ generated by the submodules $[V_i, V_j]$ with $i \neq j$.*

For $i \in \{1, 2, 3\}$, let V_i be the $\mathbb{Z}$-module spanned by the set $\mathcal{V}_i = \{x_{2i-1}, x_{2i}\}$. Note that $\mathcal{X} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$. Thus $L = L(V_1 \oplus V_2 \oplus V_3)$. By Corollary 1, we have

$$L = \Big(\bigoplus_{i=1}^{3} L(V_i)\Big) \oplus L(W),$$

where $W = \bigoplus_{n \geq 2} W_n$ and

$$W_n = \bigoplus_{\substack{i_1 > i_2 \leq i_3 \leq \cdots \leq i_n \\ i_1, \ldots, i_n \in \{1,2,3\}}} [V_{i_1}, V_{i_2}, \ldots, V_{i_n}].$$

Note that $L(W)$ is an ideal of L. For $n \geq 2$, we write

$$\begin{aligned} W_n \quad = \quad & \Big(\bigoplus\nolimits_{\substack{\alpha+\beta+\gamma=n-2 \\ \alpha,\beta,\gamma \geq 0}} [V_2, V_1, \ {}_{\alpha}V_1, \ {}_{\beta}V_2, \ {}_{\gamma}V_3]\Big) \ \oplus \\ & \Big(\bigoplus\nolimits_{\substack{\alpha_1+\beta_1+\gamma_1=n-2 \\ \alpha_1,\beta_1,\gamma_1 \geq 0}} [V_3, V_1, \ {}_{\alpha_1}V_1, \ {}_{\beta_1}V_2, \ {}_{\gamma_1}V_3]\Big) \ \oplus \\ & \Big(\bigoplus\nolimits_{\substack{\delta+\epsilon=n-2 \\ \delta,\epsilon \geq 0}} [V_3, V_2, \ {}_{\delta}V_2, \ {}_{\epsilon}V_3]\Big). \end{aligned}$$

Write

$$[V_2, V_1] = V_{211} \oplus V_{212}, \quad [V_3, V_1] = V_{311} \oplus V_{312} \text{ and } [V_3, V_2] = V_{321} \oplus V_{322},$$

where $V_{211} = \langle y_4, y_5 \rangle$, $V_{212} = \langle y_6, y_7 \rangle$, $V_{311} = \langle y_8, y_{11} \rangle$, $V_{312} = \langle y_9, y_{10} \rangle$, $V_{321} = \langle y_{13}, y_{14} \rangle$ and $V_{322} = \langle y_{12}, y_{15} \rangle$. Furthermore, we denote

$$\begin{aligned} W_n^{(2,1)} = \quad & (\textstyle\bigoplus_{\substack{\alpha+\beta+\gamma=n-2 \\ \alpha \geq 1}} ([V_{211},\ {}_\alpha V_1,\ {}_\beta V_2,\ {}_\gamma V_3] \oplus [V_{212},\ {}_\alpha V_1,\ {}_\beta V_2,\ {}_\gamma V_3])) \oplus \\ & (\textstyle\bigoplus_{\substack{\beta_1+\gamma_1=n-2 \\ \beta_1 \geq 1}} ([V_{211},\ {}_{\beta_1} V_2,\ {}_{\gamma_1} V_3] \oplus [V_{212},\ {}_{\beta_1} V_2,\ {}_{\gamma_1} V_3])) \oplus \\ & ([V_{211},\ {}_{(n-2)} V_3] \oplus [V_{212},\ {}_{(n-2)} V_3]), \\ W_n^{(3,1)} = \quad & (\textstyle\bigoplus_{\substack{\alpha+\beta+\gamma=n-2 \\ \alpha \geq 1}} ([V_{311},\ {}_\alpha V_1,\ {}_\beta V_2,\ {}_\gamma V_3] \oplus [V_{312},\ {}_\alpha V_1,\ {}_\beta V_2,\ {}_\gamma V_3])) \oplus \\ & (\textstyle\bigoplus_{\substack{\beta_1+\gamma_1=n-2 \\ \beta_1 \geq 1}} ([V_{311},\ {}_{\beta_1} V_2,\ {}_{\gamma_1} V_3] \oplus [V_{312},\ {}_{\beta_1} V_2,\ {}_{\gamma_1} V_3])) \oplus \\ & ([V_{311},\ {}_{(n-2)} V_3] \oplus [V_{312},\ {}_{(n-2)} V_3]), \end{aligned}$$

and

$$\begin{aligned} W_n^{(3,2)} = \quad & (\textstyle\bigoplus_{\substack{\beta+\gamma=n-2 \\ \beta \geq 1}} ([V_{323},\ {}_\beta V_2,\ {}_\gamma V_3] \oplus [V_{324},\ {}_\beta V_2,\ {}_\gamma V_3])) \oplus \\ & ([V_{323},\ {}_{(n-2)} V_3] \oplus [V_{324},\ {}_{(n-2)} V_3]). \end{aligned}$$

Thus, for any $n \geq 2$,

$$W_n = W_n^{(2,1)} \oplus W_n^{(3,1)} \oplus W_n^{(3,2)}.$$

For $(j, i) \in \{(2, 1), (3, 1), (3, 2)\}$, let $\mathcal{W}_n^{(j,i)}$ denote the natural $\mathbb{Z}$-basis of $W_n^{(j,i)}$. Thus

$$\mathcal{W}_n = \mathcal{W}_n^{(2,1)} \cup \mathcal{W}_n^{(3,1)} \cup \mathcal{W}_n^{(3,2)}$$

is a $\mathbb{Z}$-basis of W_n. Furthermore, by $\mathcal{W}$, we write for the disjoint union of all $\mathcal{W}_n$ with $n \geq 2$

$$\mathcal{W} = \bigcup_{n \geq 2} \mathcal{W}_n$$

which is a $\mathbb{Z}$-basis of W. For the elements of $\mathcal{W}_n$, $n \geq 2$, and so, for the elements of $\mathcal{W}$, we introduce the following notation: Let α, β, γ be non-negative integers with $\alpha + \beta + \gamma = n - 2$, $n \geq 2$. For $\mu \in \{3, 4, 5, 6\}$, $\nu \in \{1, 2\}$, let

$$v_{n,(\alpha,\beta,\gamma)}^{(\mu,\nu)} = [x_\mu, x_\nu, x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}]$$

with $x_{1,1}, \ldots, x_{\alpha,1} \in \mathcal{V}_1$, $x_{1,2}, \ldots, x_{\beta,2} \in \mathcal{V}_2$, $x_{1,3}, \ldots, x_{\gamma,3} \in \mathcal{V}_3$, and for $\lambda \in \{5, 6\}$, $\tau \in \{3, 4\}$, let

$$u_{n,(\delta,\epsilon)}^{(\lambda,\tau)} = [x_\lambda, x_\tau, x_{1,2}, \ldots, x_{\delta,2}, x_{1,3}, \ldots, x_{\epsilon,3}]$$

with $x_{1,2}, \ldots, x_{\delta,2} \in \mathcal{V}_2$, $x_{1,3}, \ldots, x_{\epsilon,3} \in \mathcal{V}_3$ and $\delta + \epsilon = n - 2$. Therefore, for $n \geq 2$, we may write

$$\mathcal{W}_n = \{v_{n,(\alpha,\beta,\gamma)}^{(\mu,\nu)}, u_{n,(\delta,\epsilon)}^{(\lambda,\tau)} : \alpha, \beta, \gamma, \delta, \epsilon \geq 0, \alpha + \beta + \gamma = \delta + \epsilon = n - 2,$$
$$\mu \in \{3, 4, 5, 6\}, \nu \in \{1, 2\}, \lambda \in \{5, 6\}, \tau \in \{3, 4\}\}.$$

2.3 A generating set for the derived algebra L'

Since $L' = \bigoplus_{m\geq 2} L^m$ and $L(W) \subseteq L'$, we have by the modular law

$$L' = (\bigoplus_{i=1}^{3} L'(V_i)) \oplus L(W).$$

For a positive integer n, with $n \geq 2$, we write

$$L^n_{\text{grad}}(W) = L^n \cap L(W).$$

That is, $L^n_{\text{grad}}(W)$ is the $\mathbb{Z}$-submodule of L^n spanned by all Lie commutators of the form $[v_1, \ldots, v_\kappa]$ with $\kappa \geq 1$, $v_i \in W_{n(i)}$ and $n(1) + \cdots + n(\kappa) = n$. Note that $L^2_{\text{grad}}(W) = W_2$ and $L^3_{\text{grad}}(W) = W_3$. It is easily verified that, for $n \geq 2$,

$$L^n = (\bigoplus_{i=1}^{3} L^n(V_i)) \oplus L^n_{\text{grad}}(W).$$

Furthermore, since W is spanned by homogeneous elements, we have

$$L(W) = \bigoplus_{n\geq 2} L^n_{\text{grad}}(W).$$

The following result is well known (see, for example, [2], [20]). It gives us a way of constructing a $\mathbb{Z}$-basis of a free Lie algebra.

Lemma 2 *Let $L(\mathcal{A})$ be a free Lie algebra on an ordered set $\mathcal{A}$. Then $L'(\mathcal{A})$ is a free Lie algebra with a free generating set $\mathcal{A}^{(1)}$*

$$\mathcal{A}^{(1)} = \{[a_{i_1}, \ldots, a_{i_\kappa}] : \kappa \geq 2, a_{i_1} > a_{i_2} \leq a_{i_3} \leq \cdots \leq a_{i_\kappa}, a_{i_1}, \ldots, a_{i_\kappa} \in \mathcal{A}\}.$$

For a positive integer d, let $L^{(d)}(\mathcal{A}) = (L^{(d-1)}(\mathcal{A}))'$ with $L^{(0)}(\mathcal{A}) = L(\mathcal{A})$ and $L^{(1)}(\mathcal{A}) = L'(\mathcal{A})$. If $\mathcal{A}^{(d)}$ is an ordered free generating set for $L^{(d)}(\mathcal{A})$, then $\mathcal{A}^{(d+1)}$ is a free generating set for $L^{(d+1)}(\mathcal{A})$ where

$$\mathcal{A}^{(d+1)} = \{[a^{(d)}_{i_1}, \ldots, a^{(d)}_{i_\kappa}] : \kappa \geq 2, a^{(d)}_{i_1} > a^{(d)}_{i_2} \leq a^{(d)}_{i_3} \leq \cdots \leq a^{(d)}_{i_\kappa}, a^{(d)}_{i_1}, \ldots, a^{(d)}_{i_\kappa} \in \mathcal{A}^{(d)}\}.$$

By Lemma 2 (for $\mathcal{A} = \mathcal{X}$), $L' = L'(\mathcal{X})$ is a free Lie algebra with a free generating set

$$\mathcal{X}^{(1)} = \{[x_{i_1}, \ldots, x_{i_r}] : r \geq 2, i_1 > i_2 \leq i_3 \leq \cdots \leq i_r, i_1, \ldots, i_r \in \{1, \ldots, 6\}\}.$$

The set $\mathcal{X}^{(1)}$ is called *the standard free generating set* of L'. For a positive integer n, with $n \geq 2$, let

$$\mathcal{X}_n = \mathcal{X}^{(1)} \cap L^n.$$

Note that $\mathcal{X}^{(1)}$ decomposes into disjoint finite subsets $\mathcal{X}_n$

$$\mathcal{X}^{(1)} = \bigcup_{n\geq 2} \mathcal{X}_n.$$

For $n \geq 2$ and $i = 1, 2, 3$, let

$$\begin{array}{lll}\mathcal{X}_{n,i} & = & \mathcal{X}_n \cap L^n(V_i) = \mathcal{X}^{(1)} \cap L^n(V_i) \\ \text{and} & & \\ \mathcal{X}_{n,W} & = & \mathcal{X}_n \cap L^n_{\text{grad}}(W) = \mathcal{X}^{(1)} \cap L^n_{\text{grad}}(W).\end{array}$$

It is easily verified that

$$\mathcal{X}_{2,i} = \{[x_{2i}, x_{2i-1}]\}, \quad \mathcal{X}_{2,W} = \{y_4, \ldots, y_{15}\}$$

and, for $n \geq 3$, $\mathcal{X}_{n,i}$ consists of all Lie commutators of the form

$$[x_{2i}, x_{2i-1}, \ {}_\alpha x_{2i-1}, \ {}_\beta x_{2i}]$$

with non-negative integers α, β and $\alpha + \beta = n - 2$. The proof of the following result is straightforward.

Lemma 3 *For $n \geq 3$, $\mathcal{X}_{n,W}$ consists of all Lie commutators of the form $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$, $i_1 > i_2 \leq i_3 \leq \cdots \leq i_n$, $i_1, \ldots, i_n \in \{1, \ldots, 6\}$ and either $(i_1, i_2) \in \{(2,1), (4,3)\}$ and at least one of $i_3, \ldots, i_n$ not in $\{i_1, i_2\}$ or $(i_1, i_2) \in \{(j, i) : 1 \leq i < j \leq 6\} \setminus \{(2,1), (4,3), (6,5)\}$.*

By the $\mathbb{Z}$-module decomposition of L^n, with $n \geq 2$, $\mathcal{X}_n$ decomposes into disjoint subsets $\mathcal{X}_{n,i}$, $i = 1, 2, 3$, and $\mathcal{X}_{n,W}$

$$\mathcal{X}_n = (\bigcup_{i=1}^{3} \mathcal{X}_{n,i}) \cup \mathcal{X}_{n,W}.$$

The elements of $\mathcal{X}_{n,i}$ $(i = 1, 2, 3)$ are arbitrarily ordered, and this order can be extended to $\mathcal{X}_n$ subject to $u < v$ if $u \in \mathcal{X}_{n,i}, v \in \mathcal{X}_{n,j}$ with $i < j$ and $u < v$ for $u \in \bigcup_{i=1}^{3} \mathcal{X}_{n,i}$ and $v \in \mathcal{X}_{n,W}$. Thus

$$\mathcal{X}^{(1)} = \bigcup_{n \geq 2} ((\bigcup_{i=1}^{3} \mathcal{X}_{n,i}) \cup \mathcal{X}_{n,W})$$

is an ordered (free) generating set for L'.

3 Changing generating sets

3.1 A new generating set for L'

It is essential, in this paper, to replace a given free generating set of a free Lie algebra by another free generating set. This can be done by using a Lie algebra automorphism. The following result was proved in [6, Lemma 2.1].

Lemma 4 *Let $\mathcal{Z}$ be a countable set, and we assume that it is decomposed into a disjoint union $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \cdots$ of finite subsets $\mathcal{Z}_i = \{z_{i,1}, \ldots, z_{i,k_i}\}$ in such a way that $z_{i,1} < \cdots < z_{i,k_i}$ and $z_{i,k_i} < z_{i+1,1}$ for all i. Let $L(\mathcal{Z})$ denote the free Lie algebra on $\mathcal{Z}$. Given $z \in \mathcal{Z}$, we let $L_{\mathcal{Z}}(< z)$ denote the free Lie subalgebra of $L(\mathcal{Z})$ that is generated by all $x \in \mathcal{Z}$ with $x < z$. For each i, let ϕ_i be any automorphism of the free $\mathbb{Z}$-module spanned by $\mathcal{Z}_i$. Let $\phi : \mathcal{Z} \longrightarrow L(\mathcal{Z})$ be the map given by $\phi(z_{i,j}) = \phi_i(z_{i,j}) + u_{i,j}$, where $u_{i,j} \in L_{\mathcal{Z}}(< z_{i,1})$. Then $\phi(\mathcal{Z})$ is a free generating set of $L(\mathcal{Z})$.*

In this section, we apply Lemma 4 to construct a different free generating set of L'. Namely, we have the following result. Let L be the free Lie algebra over $\mathbb{Z}$ of rank 6, freely generated by the ordered set $\mathcal{X} = \{x_1, \ldots, x_6\}$ with $x_1 < x_2 < \cdots < x_6$. For $n \geq 2$, let the map $\psi_n : \mathcal{X}_n \longrightarrow L'$ be defined as follows: For $n = 2$,

$$\begin{array}{rcl} \psi_2(y_i) & = & y_i, i = 1, \ldots, 5, 8, 11, 13, 14, \\ \psi_2(y_6) & = & y_6 + y_7, \\ \psi_2(y_7) & = & y_7 + y_5, \\ \psi_2(y_9) & = & y_9 + y_8, \\ \psi_2(y_{10}) & = & y_{10} + y_8, \\ \psi_2(y_{12}) & = & y_{12} + y_{13}, \\ \psi_2(y_{15}) & = & y_{15} + y_{13}, \end{array}$$

and, for $n \geq 3$,

$$\psi_n([x_{i_1}, \ldots, x_{i_n}]) \quad = \quad [\psi_2([x_{i_1}, x_{i_2}]), x_{i_3}, \ldots, x_{i_n}]$$

with $i_1 > i_2 \leq i_3 \leq \cdots \leq i_n$, $i_1, \ldots, i_n \in \{1, \ldots, 6\}$. Then ψ_2 extends to an automorphism of L^2 and so, $\psi_2(\mathcal{X}_2)$ is a $\mathbb{Z}$-basis of L^2. By the definition of ψ_n and since $\mathcal{X}^{(1)}$ is a free generating set for L', there exists an automorphism ϕ_n of the free $\mathbb{Z}$-module spanned by $\mathcal{X}_n$ such that

$$\psi_n([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]) = [\psi_2([x_{i_1}, x_{i_2}]), x_{i_3}, \ldots, x_{i_n}] = \phi_n([x_{i_1}, \ldots, x_{i_n}]) + u_{i_1,\ldots,i_n},$$

where $u_{i_1,\ldots,i_n} \in L_{\mathcal{X}^{(1)}}(< [x_{i_1}, \ldots, x_{i_n}])$. We define the map $\Psi : \mathcal{X}^{(1)} \longrightarrow L'$ by

$$\Psi([x_{i_1}, \ldots, x_{i_n}]) = \psi_n([x_{i_1}, \ldots, x_{i_n}])$$

with $i_1 > i_2 \leq i_3 \leq \cdots \leq i_n$ and $i_1, \ldots, i_n \in \{1, \ldots, 6\}$. The map Ψ satisfies the conditions of Lemma 4 and so, Ψ is an automorphism of L', and $\Psi(\mathcal{X}^{(1)})$ is a free generating set of L'.

3.2 A description of $\Psi(L(W))$

In this section, we give a suitable, for our purposes, description of the free Lie algebra $\Psi(L(W))$. For $n \geq 2$, we let $\mathcal{X}_{n,\Psi} = \Psi(\mathcal{X}_n)$, that is,

$$\mathcal{X}_{n,\Psi} = \Psi(\mathcal{X}_n) = (\bigcup_{i=1}^{3} \mathcal{X}_{n,i}) \cup \Psi(\mathcal{X}_{n,W}).$$

By Lemma 3 and the definition of Ψ, we have $\Psi(\mathcal{X}_{n,W}) (= \mathcal{X}_{n,\Psi,W})$ consists of all Lie commutators of the form $[\psi_2([x_{i_1}, x_{i_2}]), x_{i_3}, \ldots, x_{i_n}]$ for all $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] \in \mathcal{X}_{n,W}$. Thus, for $n \geq 2$, $\mathcal{X}_{n,\Psi}$ decomposes into disjoint subsets $\mathcal{X}_{n,i}$, $i = 1, 2, 3$, and $\mathcal{X}_{n,\Psi,W}$

$$\mathcal{X}_{n,\Psi} = (\bigcup_{i=1}^{3} \mathcal{X}_{n,i}) \cup \mathcal{X}_{n,\Psi,W}.$$

The elements of $\mathcal{X}_{n,i}$ $(i = 1, 2, 3)$ and $\mathcal{X}_{n,\Psi,W}$ are arbitrarily ordered, and extend it to $\mathcal{X}_{n,\Psi}$ subject to $u < v$ if $u \in \mathcal{X}_{n,r}, v \in \mathcal{X}_{n,s}$ with $r < s$ and for $u \in \bigcup_{i=1}^{3} \mathcal{X}_{n,i}$ and $v \in \mathcal{X}_{n,\Psi,W}$ we have $u < v$. For $\kappa \in \{1, 2, 3\}$, we write

$$\mathcal{X}_{(\kappa,\Psi)} = \bigcup_{n \geq 2} \mathcal{X}_{n,\kappa} \quad \text{and} \quad \mathcal{X}_{(4,\Psi)} = \bigcup_{n \geq 2} \mathcal{X}_{n,\Psi,W}.$$

Note that $\Psi(\mathcal{X}^{(1)}) = \bigcup_{\kappa=1}^{4} \mathcal{X}_{(\kappa,\Psi)}$.

For $\kappa = 1, \ldots, 4$, we let $U_{(\kappa,\Psi)}$ be the $\mathbb{Z}$-module spanned by the set $\mathcal{X}_{(\kappa,\Psi)}$. Since L' is free on $\Psi(\mathcal{X}^{(1)})$, and $\Psi(\mathcal{X}^{(1)})$ is a $\mathbb{Z}$-basis of $U_{(1,\Psi)} \oplus \cdots \oplus U_{(4,\Psi)}$, we have L' is free on $U_{(1,\Psi)} \oplus \cdots \oplus U_{(4,\Psi)}$ and so, by Corollary 1,

$$L' \quad = L(U_{(1,\Psi)}) \oplus \cdots \oplus L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)}),$$

where $W^{(1,\ldots,4,\Psi)} = \bigoplus_{n\geq 2} W_n^{(1,\ldots,4,\Psi)}$ and, for $n \geq 2$,

$$W_n^{(1,\ldots,4,\Psi)} = \bigoplus_{\substack{i_1 > i_2 \leq i_3 \leq \cdots \leq i_n \\ i_1,\ldots,i_n \in \{1,2,3,4\}}} [U_{(i_1,\Psi)}, U_{(i_2,\Psi)}, \ldots, U_{(i_n,\Psi)}].$$

Furthermore, by Corollary 1, $L(W^{(1,\ldots,4,\Psi)})$ is an ideal of L'. Since $L(U_{(i,\Psi)}) = L'(V_i)$, $i = 1, 2, 3$, we have

$$L' = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)}).$$

It is clear that $L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$ is a Lie subalgebra of L'. Our aim in this subsection is to show the following result.

Proposition 1 *Let L be a free Lie algebra over $\mathbb{Z}$ of rank 6, freely generated by the ordered set $\mathcal{X} = \{x_1, \ldots, x_6\}$ with $x_1 < x_2 < \cdots < x_6$. Let Ψ be the automorphism of L' defined in subsection 3.1. Then*

$$\Psi(L(W)) = L(\Psi(W)) = L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$$

as $\mathbb{Z}$-modules.

For the proof of Proposition 1, we need some extra notation and technical results. Write

$$\begin{array}{rcl}
\Psi([V_2, V_1]) & = & [V_2, V_1]^{(1)} \oplus [V_2, V_1]^{(2)}, \\
\Psi([V_3, V_1]) & = & [V_3, V_1]^{(1)} \oplus [V_3, V_1]^{(2)}, \\
\Psi([V_3, V_2]) & = & [V_3, V_2]^{(1)} \oplus [V_3, V_2]^{(2)},
\end{array}$$

where

$$[V_2, V_1]^{(1)} = \langle y_4, y_5 \rangle, \quad [V_2, V_1]^{(2)} = \langle \psi_2(y_6), \psi_2(y_7) \rangle,$$

$$[V_3, V_1]^{(1)} = \langle y_8, y_{11} \rangle, \quad [V_3, V_1]^{(2)} = \langle \psi_2(y_9), \psi_2(y_{10}) \rangle,$$

and

$$[V_3, V_2]^{(1)} = \langle y_{13}, y_{14} \rangle, \quad [V_3, V_2]^{(2)} = \langle \psi_2(y_{12}), \psi_2(y_{15}) \rangle.$$

Thus

$$\Psi(W_2)(= W_{2,\Psi}) = W_{2,\Psi}^{(1)} \oplus W_{2,\Psi}^{(2)},$$

where, for $i = 1, 2$, $W_{2,\Psi}^{(i)} = [V_2, V_1]^{(i)} \oplus [V_3, V_1]^{(i)} \oplus [V_3, V_2]^{(i)}$. For a positive integer n, with $n \geq 3$, non-negative integers α, β, γ with $\alpha + \beta + \gamma = n - 2$, $(j, i) \in \{(2, 1), (3, 1)\}$ and $\mu = 1, 2$, we put

$$W_{n,\Psi,(\alpha,\beta,\gamma)}^{(j,i,\mu)} = [[V_j, V_i]^{(\mu)}, \ {}_{\alpha}V_1, \ {}_{\beta}V_2, \ {}_{\gamma}V_3]$$

and for non-negative integers δ, ϵ with $\delta + \epsilon = n - 2$

$$W^{(3,2,\mu)}_{n,\Psi,(\delta,\epsilon)} = [[V_3, V_2]^{(\mu)},\ {}_\delta V_2,\ {}_\epsilon V_3].$$

For $(j,i) \in \{(2,1),(3,1)\}$, let

$$W^{(j,i)}_{n,\Psi,(\alpha,\beta,\gamma)} = W^{(j,i,1)}_{n,\Psi,(\alpha,\beta,\gamma)} + W^{(j,i,2)}_{n,\Psi,(\alpha,\beta,\gamma)}$$

and

$$W^{(3,2)}_{n,\Psi,(\delta,\epsilon)} = W^{(3,2,1)}_{n,\Psi,(\delta,\epsilon)} + W^{(3,2,2)}_{n,\Psi,(\delta,\epsilon)}.$$

Since $\mathcal{W} = \bigcup_{n\geq 2} \mathcal{W}_n$ is a $\mathbb{Z}$-basis of W and since Ψ is an automorphism of L', we have the above sums are direct, and further, the sum of $W^{(j,i)}_{n,\Psi,(\alpha,\beta,\gamma)}$ over all 3-tuples (α,β,γ) with $\alpha+\beta+\gamma = n-2$ is direct denoted by $W^{(j,i)}_{n,\Psi}$. Our next technical result shows that

$$\sum_{\substack{\delta+\epsilon=n-2\\ \delta,\ \epsilon\geq 0}} W^{(3,2)}_{n,\Psi,(\delta,\epsilon)}$$

is in fact direct. By the action of ψ_2 on $\mathcal{X}_2$, $\mathcal{W}_n$ is a part of a free generating set and since $\mathcal{X}^{(1)}$ is a free generating of L', we have the following result.

Lemma 5 *Let n be a positive integer, with $n \geq 3$. Then, the sum of $W^{(3,2)}_{n,\Psi,(\delta,\epsilon)}$ over all 2-tuples (δ,ϵ) of non-negative integers δ,ϵ with $\delta+\epsilon = n-2$ is direct, denoted by $W^{(3,2)}_{n,\Psi}$.*

By the definition of Ψ, we have $W^{(2,1)}_{n,\Psi} + W^{(3,1)}_{n,\Psi} + W^{(3,2)}_{n,\Psi}$, $n \geq 2$, is direct, denoted by $W_{n,\Psi}$. For $n \geq 2$ and $i = 1,2$, we write

$$W^{(i)}_{n,\Psi} = W^{(2,1,i)}_{n,\Psi} \oplus W^{(3,1,i)}_{n,\Psi} \oplus W^{(3,2,i)}_{n,\Psi}.$$

Since $W^{(i)}_{\Psi} = \sum_{n\geq 2} W^{(i)}_{n,\Psi}$, $i = 1,2$, is direct, and $W^{(1)}_{\Psi} \cap W^{(2)}_{\Psi} = \{0\}$, we denote

$$W_{\Psi} = W^{(1)}_{\Psi} \oplus W^{(2)}_{\Psi}.$$

For $n \geq 2$ and $i = 1,2$, let $\mathcal{W}^{(i)}_{n,\Psi}$ denote the natural $\mathbb{Z}$-basis of $W^{(i)}_{n,\Psi}$. More precisely, let α,β,γ be nonnegative integers with $\alpha+\beta+\gamma = n-2$, $n \geq 2$. For $\mu \in \{3,4,5,6\}$, $\nu \in \{1,2\}$, we define

$$v^{(\mu,\nu,\Psi)}_{n,(\alpha,\beta,\gamma)} = [\psi_2([x_\mu, x_\nu]), x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}]$$

with $x_{1,1},\ldots,x_{\alpha,1} \in \mathcal{V}_1$, $x_{1,2},\ldots,x_{\beta,2} \in \mathcal{V}_2$, $x_{1,3},\ldots,x_{\gamma,3} \in \mathcal{V}_3$, and for $\lambda \in \{5,6\}$, $\tau \in \{3,4\}$,

$$u^{(\lambda,\tau,\Psi)}_{n,(\delta,\epsilon)} = [\psi_2([x_\lambda, x_\tau]), x_{1,2}, \ldots, x_{\delta,2}, x_{1,3}, \ldots, x_{\epsilon,3}]$$

with $x_{1,2},\ldots,x_{\delta,2} \in \mathcal{V}_2$, $x_{1,3},\ldots,x_{\epsilon,3} \in \mathcal{V}_3$ and $\delta+\epsilon = n-2$. Thus, for $n \geq 2$,

$\mathcal{W}^{(1)}_{n,\Psi} = \{v^{(3,1,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(4,1,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(5,1,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(6,2,\Psi)}_{n,(\alpha,\beta,\gamma)},$

$$u^{(6,3,\Psi)}_{n,(\delta,\epsilon)}, u^{(5,4,\Psi)}_{n,(\delta,\epsilon)} : \alpha,\beta,\gamma,\delta,\epsilon \geq 0, \alpha+\beta+\gamma = \delta+\epsilon = n-2\}$$

and

$$\mathcal{W}^{(2)}_{n,\Psi} = \{v^{(3,2,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(4,2,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(6,1,\Psi)}_{n,(\alpha,\beta,\gamma)}, v^{(5,2,\Psi)}_{n,(\alpha,\beta,\gamma)},$$

$$u^{(5,3,\Psi)}_{n,(\delta,\epsilon)}, u^{(6,4,\Psi)}_{n,(\delta,\epsilon)} : \alpha,\beta,\gamma,\delta,\epsilon \geq 0, \alpha+\beta+\gamma = \delta+\epsilon = n-2\}.$$

Furthermore, we write $\mathcal{W}^{(i)}_{\Psi} = \bigcup_{n\geq 2} \mathcal{W}^{(i)}_{n,\Psi}$ $(i = 1, 2)$, which is the natural $\mathbb{Z}$-basis of $W^{(i)}_{\Psi}$ and so, $\mathcal{W}_{\Psi} = \mathcal{W}^{(1)}_{\Psi} \cup \mathcal{W}^{(2)}_{\Psi}$ is a $\mathbb{Z}$-basis of W_{Ψ}. We note that $W^{(2)}_{\Psi} \subseteq J$. For $n \geq 2$, we write $\mathcal{W}_{n,\Psi} = \mathcal{W}^{(1)}_{n,\Psi} \cup \mathcal{W}^{(2)}_{n,\Psi}$ and so, $\mathcal{W}_{\Psi} = \bigcup_{n\geq 2} \mathcal{W}_{n,\Psi}$.

Lemma 6 *Let $L(\mathcal{W}_{\Psi})$ be the Lie subalgebra of L' generated by $\mathcal{W}_{\Psi}$. Then $L(\mathcal{W}_{\Psi}) = L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$.*

Proof. Recall that

$$L' = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)}).$$

We shall prove our claim into two steps.

Step 1. $L(\mathcal{W}_{\Psi}) \subseteq L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$

Since $L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$ is a Lie subalgebra of L', it is enough to show that

$$\mathcal{W}_{\Psi} \subseteq L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)}).$$

Since $\mathcal{W}_{\Psi} = \bigcup_{n\geq 2} \mathcal{W}_{n,\Psi}$, it is enough to show that, for any $n \geq 2$,

$$\mathcal{W}_{n,\Psi} \subseteq L(U^{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)}).$$

For $n = 2, 3$, it is clear that $\mathcal{W}_{n,\Psi} \subseteq L(U^{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$. Thus we assume that $n \geq 4$. Since $L' = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$, by the definition of Ψ, and since $L(W^{(1,\ldots,4,\Psi)})$ is an ideal of L', we have each $v^{(\mu,\nu,\Psi)}_{n,(\alpha,\beta,\gamma)}, u^{(\lambda,\tau,\Psi)}_{n,(\delta,\epsilon)}$ belongs to $L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})$ and so, we obtain the desired result.

Step 2. $L' = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(\mathcal{W}_{\Psi})$

By Step 1,

$$(\bigoplus_{i=1}^{3} L'(V_i)) \bigcap L(\mathcal{W}_{\Psi}) = \{0\}.$$

Put

$$L'_{\Psi} = (\bigoplus_{i=1}^{3} L'(V_i)) \oplus L(\mathcal{W}_{\Psi}).$$

We claim that $L' = L'_{\Psi}$. It is enough to show that $L' \subseteq L'_{\Psi}$. By Corollary 1,

$$L = (\bigoplus_{i=1}^{3} L(V_i)) \oplus L(W),$$

and since $L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(W) \subseteq L'$ and $(V_1 \oplus V_2 \oplus V_3) \cap L' = \{0\}$, we have, by using the modular law,

$$L' = (\bigoplus_{i=1}^{3} L'(V_i)) \oplus L(W).$$

We shall show that $W \subseteq L(\mathcal{W}_\Psi)$. Since $L(\mathcal{W}_\Psi)$ is a $\mathbb{Z}$-module, and $\mathcal{W} = \bigcup_{n\geq 2} \mathcal{W}_n$ is a $\mathbb{Z}$-basis of W, it is enough to show that

$$\mathcal{W}_n \subseteq L(\mathcal{W}_\Psi)$$

for all $n \geq 2$. Furthermore, it is enough to show that

$$v^{(\mu,\nu)}_{n,(\alpha,\beta,\gamma)}, u^{(\lambda,\tau)}_{n,(\delta,\epsilon)} \in L(\mathcal{W}_\Psi)$$

for all non-negative integers $\alpha, \beta, \ldots, \epsilon \geq 0, \alpha + \beta + \gamma = \delta + \epsilon = n - 2$, $\mu \in \{3, 4, 5, 6\}, \nu \in \{1, 2\}, \lambda \in \{5, 6\}, \tau \in \{3, 4\}$. Recall that

$$[x_3, x_2] = y_6 = \psi_2(y_6) - y_7 = \psi_2(y_6) - \psi_2(y_7) + y_5,$$

$$[x_4, x_2] = y_7 = \psi_2(y_7) - y_5,$$

$$[x_6, x_1] = y_9 = \psi_2(y_9) - y_8,$$

$$[x_5, x_2] = y_{10} = \psi_2(y_{10}) - y_8,$$

$$[x_5, x_3] = y_{12} = \psi_2(y_{12}) - y_{13}$$

and

$$[x_6, x_4] = y_{15} = \psi_2(y_{15}) - y_{13}.$$

Write $\mathcal{H} = \{(\mu, \nu) : \mu \in \{3, 4, 5, 6\}, \nu \in \{1, 2\}\}$. If $(\mu, \nu) \in \mathcal{H} \setminus \{(3, 2), (4, 2), (5, 2), (6, 1)\}$, then $v^{(\mu,\nu)}_{n,(\alpha,\beta,\gamma)} \in W_{n,\Psi}$ for all non-negative integers α, β, γ with $\alpha + \beta + \gamma = n - 2$. Suppose that $(\mu, \nu) = (3, 2)$. Then

$$\begin{aligned} v^{(3,2)}_{n,(\alpha,\beta,\gamma)} &= [y_6, x_{1,1}, \ldots, x_{\alpha,1}, x_1, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}] \\ &= [\psi_2(y_6), x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}] - \\ &\quad [\psi_2(y_6), x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}] + \\ &\quad [y_5, x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}] \end{aligned}$$

where $x_{1,1}, \ldots, x_{\alpha,1} \in \mathcal{V}_1$, $x_{1,2}, \ldots, x_{\beta,2} \in \mathcal{V}_2$, $x_{1,3}, \ldots, x_{\gamma,3} \in \mathcal{V}_3$ and so, $v^{(6,1)}_{n,(\alpha,\beta,\gamma)} \in W_{n,\Psi}$ for all α, β, γ. By using similar arguments as before, we have $v^{(4,2)}_{n,(\alpha,\beta,\gamma)}, v^{(5,2)}_{n,(\alpha,\beta,\gamma)}$ and $v^{(6,1)}_{n,(\alpha,\beta,\gamma)} \in W_{n,\Psi}$. Therefore, for $(\mu, \nu) \in \mathcal{H}$, $v^{(\mu,\nu)}_{n,(\alpha,\beta,\gamma)} \in W_{n,\Psi}$ for all α, β, γ. It is easily verified that, for $(\lambda, \tau) \in \{(5, 4), (6, 3)\}$, $u^{(\lambda,\tau)}_{n,(\delta,\epsilon)} \in W_{n,\Psi}$ for all δ, ϵ. Thus, we concentrate on the cases $(5, 3)$ and $(6, 4)$. Then

$$\begin{aligned} u^{(5,3)}_{n,(\delta,\epsilon)} &= [y_{12}, x_{1,2}, \ldots, x_{\delta,2}, x_{1,3}, \ldots, x_{\epsilon,3}] \\ &= [\psi_2(y_{12}), x_{1,2}, \ldots, x_{\delta,2}, x_{1,3}, \ldots, x_{\epsilon,3}] - \\ &\quad [y_{13}, , x_{1,2}, \ldots, x_{\delta,2}, x_{1,3}, \ldots, x_{\epsilon,3}], \end{aligned}$$

where $x_{1,2}, \ldots, x_{\delta,2} \in \mathcal{V}_2, x_{1,3}, \ldots, x_{\epsilon,3} \in \mathcal{V}_3$. Similarly, $u^{(6,4)}_{n,(\delta,\epsilon)} \in W_{n,\Psi}$ for all δ and ϵ. Therefore, for all $n \geq 2$, $\mathcal{W}_n \subseteq L(\mathcal{W}_\Psi)$. Since $L(W)$ is generated by $\mathcal{W}$, we have $L(W) \subseteq L'_\Psi$ and so, $L' = L'_\Psi$. Thus

$$\begin{array}{rcl}(\text{by Step 1})\ \ L(W_\Psi) & = & L'_\Psi \cap (L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})) \\ & = & L' \cap (L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})) \\ & = & L(U_{(4,\Psi)}) \oplus L(W^{(1,\ldots,4,\Psi)})\end{array}$$

and so, we obtain the required result.

Lemma 7 *Let $L(\mathcal{W}_\Psi)$ be the Lie subalgebra of L' generated by $\mathcal{W}_\Psi$. Then $\Psi(L(W)) = L(\Psi(W)) = L(\mathcal{W}_\Psi)$. In particular, $L(\mathcal{W}_\Psi)$ is free on $\mathcal{W}_\Psi$, and it is an ideal in L'.*

Proof. We first show that $\Psi(L(W)) \subseteq L(\mathcal{W}_\Psi)$. Let $\mathcal{W}$ be the natural $\mathbb{Z}$-basis of W. Recall that $\mathcal{W} = \bigcup_{n \geq 2} \mathcal{W}_n$, where $\mathcal{W}_n$ is the $\mathbb{Z}$-basis of W_n and its elements are denoted by $v^{(\mu,\nu)}_{n,(\alpha,\beta,\gamma)}, u^{(\lambda,\tau)}_{n,(\delta,\epsilon)}$. Since $\Psi(\mathcal{W}) = \bigcup_{n \geq 2} \Psi(\mathcal{W}_n)$, it is enough to show that $\Psi(v^{(\mu,\nu)}_{n,(\alpha,\beta,\gamma)}), \Psi(u^{(\lambda,\tau)}_{n,(\delta,\epsilon)}) \in L(W_\Psi)$. By Lemma 2 (for $\mathcal{A} = \mathcal{X}$), $\mathcal{X}^{(1)}$ is a free generating set for L'. Recall that $\mathcal{X}^{(1)}$ is an ordered set. Furthermore, for a positive integer d, if $\mathcal{X}^{(d)}$ is an ordered free generating set for $L^{(d)}(\mathcal{X})$, with $L^{(1)}(\mathcal{X}) = L'$, then $\mathcal{X}^{(d+1)}$ is a free generating set for $L^{(d+1)}(\mathcal{X})$ where $\mathcal{X}^{(d+1)} = \{[x^{(d)}_{i_1}, \ldots, x^{(d)}_{i_\kappa}] : \kappa \geq 2, x^{(d)}_{i_1} > x^{(d)}_{i_2} \leq x^{(d)}_{i_3} \leq \cdots \leq x^{(d)}_{i_\kappa}, x^{(d)}_{i_1}, \ldots, x^{(d)}_{i_\kappa} \in \mathcal{X}^{(d)}\}$. Write $X^{(d)}$ for the $\mathbb{Z}$-module spanned by $\mathcal{X}^{(d)}$. Note that $L^{(d)}(\mathcal{X})/L^{(d+1)}(\mathcal{X})$ is isomorphic to $X^{(d)}$ as $\mathbb{Z}$-module. Furthermore, for a positive integer n, with $n \geq 2$, there exists a (unique) positive integer $m(n) \geq 2$ such that $L^n = \bigoplus_{j=1}^{m(n)} (L^n \bigcap X^{(j)})$. Thus, any non-zero simple Lie commutator $w = [x_{i_1}, \ldots, x_{i_n}]$, with $x_{i_1}, \ldots, x_{i_n} \in \mathcal{X}$, is written as $w = \sum_{j=1}^{m(n)} u_j$, where $u_j \in L^n \cap X^{(j)}$, $j = 1, \ldots, m(n)$. (The number of occurrence of each $x_i \in \mathcal{X}$ in w is the same in each u_j.) Thus $\Psi(w) = \sum_{j=1}^{m(n)} \Psi(u_j)$. By the definition of Ψ, we have $\Psi(u_j) \in L(\mathcal{W}_\Psi)$ for $j = 1, \ldots, m(n)$ and so, $\Psi(L(W)) \subseteq L(\mathcal{W}_\Psi)$. Since $L' = (\bigoplus_{i=1}^{3} L'(V_i)) \oplus L(W)$, by the definition of Ψ, and Ψ is an automorphism of L', we have

$$L' = (\bigoplus_{i=1}^{3} L'(V_i)) \oplus \Psi(L(W)).$$

Since $\Psi(L(W)) \subseteq L(\mathcal{W}_\Psi)$, the modular law and Lemma 6, we get

$$\Psi(L(W)) = L' \cap L(\mathcal{W}_\Psi) = L(\mathcal{W}_\Psi).$$

Proof of Proposition 1. Since $L(W)$ is a free Lie algebra on W, $L(W)$ is an ideal of L' and Ψ is an automorphism of L', we have $\Psi(L(W))$ is a free Lie algebra, and $\Psi(L(W)) \subseteq L'$. By Lemma 7 and Lemma 6, we obtain the desired result. □

3.3 An ordering on $\mathcal{W}_\Psi$

The set $\mathcal{W}_\Psi$, defined in section 3.2, will play a significant role in the proof of our main result (Theorem 2). We need to introduce a specific order on its elements. Let A be a finite totally ordered alphabet. We order the free monoid A^* with alphabetical order, that is, $u < v$ if and only if either $v = ux$ for some $x \in A^+$ (the free semigroup on A), or $u = xau'$, $v = xbv'$

for some words x, u', v' and some $a, b \in A$ with $a < b$. (Note that the empty word in A^* is regarded the smallest element in A^*.)We order the elements of the natural $\mathbb{Z}$-basis of $W_{2,\Psi}$ as follows:

$$y_4 \ll y_5 \ll y_8 \ll y_{11} \ll y_{13} \ll y_{14} \ll$$

$$\psi_2(y_6) \ll \psi_2(y_7) \ll \psi_2(y_9) \ll \psi_2(y_{10}) \ll \psi_2(y_{12}) \ll \psi_2(y_{15}).$$

Let $\mathcal{V}_t^*(t = 1, 2, 3)$ be the free monoid on $\mathcal{V}_t$, and fix a positive integer n, with $n \geq 3$. Recall that, for $\mu \in \{3, 4, 5, 6\}$, $\nu \in \{1, 2\}$,

$$v_{n,(\alpha,\beta,\gamma)}^{(\mu,\nu,\Psi)} = [\psi_2([x_\mu, x_\nu]), x_{1,1}, \ldots, x_{\alpha,1}, x_{1,2}, \ldots, x_{\beta,2}, x_{1,3}, \ldots, x_{\gamma,3}]$$

with $x_{1,1}, \ldots, x_{\alpha,1} \in \mathcal{V}_1$, $x_{1,2}, \ldots, x_{\beta,2} \in \mathcal{V}_2$, $x_{1,3}, \ldots, x_{\gamma,3} \in \mathcal{V}_3$, and for $\lambda \in \{5, 6\}$, $\tau \in \{3, 4\}$,

$$u_{n,(\delta,\epsilon)}^{(\lambda,\tau,\Psi)} = [\psi_2([x_\lambda, x_\tau]), z_{1,2}, \ldots, z_{\delta,2}, z_{1,3}, \ldots, z_{\epsilon,3}]$$

with $z_{1,2}, \ldots, z_{\delta,2} \in \mathcal{V}_2$, $z_{1,3}, \ldots, z_{\epsilon,3} \in \mathcal{V}_3$ and $\delta + \epsilon = n - 2$. For such $v_{n,(\alpha,\beta,\gamma)}^{(\mu,\nu,\Psi)}, u_{n,(\delta,\epsilon)}^{(\lambda,\tau,\Psi)}$, we write $x_{n,\alpha}^{(\mu,\nu)} = x_{1,1} \cdots x_{\alpha,1} \in \mathcal{V}_1^*$, $x_{n,\beta}^{(\mu,\nu)} = x_{1,2} \cdots x_{\beta,2}, z_{n,\delta}^{(\lambda,\tau)} = z_{1,2} \cdots z_{\delta,2} \in \mathcal{V}_2^*$ and $x_{n,\gamma}^{(\mu,\nu)} = x_{1,3} \cdots x_{\gamma,3}, z_{n,\epsilon}^{(\lambda,\tau)} = z_{1,3} \cdots z_{\epsilon,3} \in \mathcal{V}_3^*$. We write $v_{n,(\alpha_1,\beta_1,\gamma_1)}^{(\mu_1,\nu_1,\Psi)} \ll v_{n,(\alpha_2,\beta_2,\gamma_2)}^{(\mu_2,\nu_2,\Psi)}$ if either $\psi_2([x_{\mu_1}, x_{\nu_1}]) \ll \psi_2([x_{\mu_2}, x_{\nu_2}])$ or, if $\psi_2([x_{\mu_1}, x_{\nu_1}]) = \psi_2([x_{\mu_2}, x_{\nu_2}])$, and $x_{n,\alpha_1}^{(\mu_1,\nu_1)} < x_{n,\alpha_2}^{(\mu_2,\nu_2)}$ or, if $\psi_2([x_{\mu_1}, x_{\nu_1}]) = \psi_2([x_{\mu_2}, x_{\nu_2}])$, $x_{n,\alpha_1}^{(\mu_1,\nu_1)} = x_{n,\alpha_2}^{(\mu_2,\nu_2)}$, and $x_{n,\beta_1}^{(\mu_1,\nu_1)} < x_{n,\beta_2}^{(\mu_2,\nu_2)}$ or, if $\psi_2([x_{\mu_1}, x_{\nu_1}]) = \psi_2([x_{\mu_2}, x_{\nu_2}])$, $x_{n,\alpha_1}^{(\mu_1,\nu_1)} = x_{n,\alpha_2}^{(\mu_2,\nu_2)}$, $x_{n,\beta_1}^{(\mu_1,\nu_1)} = x_{n,\beta_2}^{(\mu_2,\nu_2)}$, and $x_{n,\gamma_1}^{(\mu_1,\nu_1)} < x_{n,\gamma_2}^{(\mu_2,\nu_2)}$. Similarly, we define $u_{n,(\delta_1,\epsilon_1)}^{(\lambda_1,\tau_1,\Psi)} < u_{n,(\delta_2,\epsilon_2)}^{(\lambda_2,\tau_2,\Psi)}$. Thus the elements of $\mathcal{W}_{n,\Psi}$ are totally ordered. Extend this ordering to $\mathcal{W}_\Psi$ by setting $u \ll v$ for all $u \in \mathcal{W}_{n,\Psi}$ and $v \in \mathcal{W}_{m,\Psi}$ with $n < m$.

4 The ideal J

4.1 Preliminaries

Recall that J is the ideal of L generated by the set $\mathcal{V}$, where $\mathcal{V} = \{y_1, y_2, y_3, y_6 + y_7, y_7 + y_5, y_9 + y_8, y_{10} + y_8, y_{12} + y_{13}, y_{15} + y_{13}\}$. For a non-negative integer m, we write $[L^2, {}_mL]$ for the $\mathbb{Z}$-module spanned by Lie commutators of the form $[v, u_1, \ldots, u_m]$ with $u_1, \ldots, u_m \in L$ and $v \in L^2$. By convention, $L^2 = [L^2, {}_0L]$. Direct calculations show that $\mathcal{V}$ is a $\mathbb{Z}$-basis of $L^2 \cap J$. Namely, we have the following $\mathbb{Z}$-module decomposition of L^2

$$L^2 = (L^2 \cap J) \oplus (L^2)^*,$$

where $(L^2)^*$ is the $\mathbb{Z}$-submodule of L^2 spanned by the set $\mathcal{V}^* = \{y_4, y_5, y_8, y_{11}, y_{13}, y_{14}\}$. Note that $(L^2)^* = W_{2,\Psi}^{(1)}$. The proof of the following result is elementary.

Lemma 8 *Let $\widetilde{J} = \sum_{m \geq 0} [L^2 \cap J, {}_mL]$. Then $\widetilde{J} = J$.*

Another useful technical result is the following.

Lemma 9 *For each $c \geq 0$, let*

$$J_c = \sum_{m \geq c} [L^2 \cap J, \ _m L].$$

Then J_c is an ideal of L for all c, $J_c = [L^2 \cap J, \ _c L^1] \oplus J_{c+1}$ and $[L^2 \cap J, \ _c L^1] = L^{c+2} \cap J = J^{c+2}$. Furthermore,

$$J_c = \bigoplus_{m \geq c} ([L^2 \cap J, \ _m L^1]).$$

Proof. It is clear that J_c is an ideal of L for all c and $J_0 = J$. Since $L^2 \cap J \subseteq L^2$ and $J_1 \subseteq \gamma_3(L)$, we have $J = (L^2 \cap J) \oplus J_1$. It is easily verified that J is the Lie subalgebra of L generated by the Lie commutators of the form $h = [y, a_1, \ldots, a_\kappa]$ where $y \in L^2 \cap J$ and $a_1, \ldots, a_\kappa \in L$. Since $L = \oplus_{m \geq 1} L^m$, each L^m is a $\mathbb{Z}$-module spanned by the Lie commutators $[x_{j_1}, \ldots, x_{j_m}]$ with $j_1, \ldots, j_m \in \{1, \ldots, 6\}$, and the Lie commutators are multi-linear operations, we may assume that each a_μ is a simple Lie commutator of the form $[x_{j_1}, \ldots, x_{j_{m(\mu)}}]$ with $j_1, \ldots, j_{m(\mu)} \in \{1, \ldots, 6\}$ and $\mu \geq 1$. Thus J is generated as a Lie subalgebra by the set

$$\{[y, [x_{j_{1,1}}, \ldots, x_{j_{m(1),1}}], \ldots, [x_{j_{1,\kappa}}, \ldots, x_{j_{m(\kappa),\kappa}}]] : m(1) + \cdots + m(\kappa) \geq 0,$$

$$[x_{j_{1,\lambda}}, \ldots, x_{j_{m(\lambda),\lambda}}] \in L^{m(\lambda)}, x_{j_{1,\lambda}}, \ldots, x_{j_{m(\lambda),\lambda}} \in \{x_1, \ldots, x_6\}, \lambda = 1, \ldots, \kappa\}.$$

For $\nu \in \{1, \ldots, \kappa\}$, let $u_\nu = [x_{j_{1,\nu}}, \ldots, x_{j_{m(\nu),\nu}}]$. Using the Jacobi identity in the form $[x, [y, z]] = [x, y, z] - [x, z, y]$, we may write each $[y, u_1, \ldots, u_\kappa]$, with $m(1) + \cdots + m(\kappa) \geq c$, as a $\mathbb{Z}$-linear combination of Lie commutators of the form

$$[y, x_{j_{1,1}}, \ldots, x_{j_{m(1),1}}, \ldots, x_{j_{1,\kappa}}, \ldots, x_{j_{m(\kappa),\kappa}}]$$

with $m(1) + \cdots + m(\kappa) \geq c$. Let $\mathcal{J}_{\geq c} = \{[y, x_{j_{1,1}}, \ldots, x_{j_{m(1),1}}, \ldots, x_{j_{1,\kappa}}, \ldots, x_{j_{m(\kappa),\kappa}}] : m(1) + \cdots + m(\kappa) \geq c, j_{1,\lambda}, \ldots, j_{m(\lambda),\lambda} \in \{1, \ldots, 6\}, \lambda = 1, \ldots, \kappa\}$. It is clearly enough that the Lie subalgebra of L generated by $\mathcal{J}_{\geq c}$ is equal to J_c. Let

$$\mathcal{J}^c_{\geq c} = \{[y, x_{j_{1,1}}, \ldots, x_{j_{m(1),1}}, \ldots, x_{j_{1,\kappa}}, \ldots, x_{j_{m(\kappa),\kappa}}] \in \mathcal{J}_{\geq c} : m(1) + \cdots + m(\kappa) = c,$$

$$j_{1,\lambda}, \ldots, j_{m(\lambda),\lambda} \in \{1, \ldots, 6\}, \lambda = 1, \ldots, \kappa\}$$

and let $J^c_{\geq c}$ be the $\mathbb{Z}$-module spanned by $\mathcal{J}^c_{\geq c}$. Thus $J^c_{\geq c} = [L^2 \cap J, \ _c L^1]$. Since $[L^2 \cap J, \ _c L^1] \subseteq L^{c+2}$ and $J_{c+1} \subseteq \gamma_{c+3}(L)$, we have

$$J_c = [L^2 \cap J, \ _c L^1] \oplus J_{c+1}$$

and so,

$$J = \bigoplus_{c \geq 0} [L^2 \cap J, \ _c L^1].$$

On the other hand, since J is generated by a set of homogenous elements, we have

$$J = \bigoplus_{m \geq 2} (L^m \cap J).$$

Therefore, for all $c \geq 0$, $[L^2 \cap J, \ _c L^1] = J \cap L^{c+2} = J^{c+2}$. $\square$

4.2 A decomposition of J

By Lemma 7, Lemma 1 and Corollary 1, we have

$$L(W_\Psi) \quad = \quad L(W_\Psi^{(1)}) \oplus L(W_\Psi^{(2)}) \oplus L(\widetilde{W}_{\Psi,J}),$$

where $\widetilde{W}_{\Psi,J} = \bigoplus_{n\geq 2} \widetilde{W}_{n,\Psi,J}$ and, for $n \geq 2$,

$$\widetilde{W}_{n,\Psi,J} = \bigoplus_{\substack{\alpha+\beta=n-1 \\ \alpha,\beta\geq 0}} [W_\Psi^{(2)},\ {}_\alpha W_\Psi^{(1)},\ {}_\beta W_\Psi^{(2)}].$$

Note that $L(\widetilde{W}_{\Psi,J})$ is an ideal in $L(W_\Psi)$. Thus we have the following $\mathbb{Z}$-module decomposition of $L(W_\Psi)$ by means of the ideal J.

Lemma 10 *The free Lie algebra $L(W_\Psi)$ decomposes (as $\mathbb{Z}$-module) into a direct sum of the free Lie algebras $L(W_\Psi^{(1)})$ and $L(W_\Psi^{(2)} \wr W_\Psi^{(1)}) = L(W_\Psi^{(2)}) \oplus L(\widetilde{W}_{\Psi,J})$. In particular, $L(W_\Psi^{(2)} \wr W_\Psi^{(1)}) \subseteq J$.*

By the proof of Lemma 6 (Step 2),

$$L' = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(W_\Psi).$$

By Lemma 10 and the modular law, we have

$$J = (L(W_\Psi^{(1)}) \cap J) \oplus L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(W_\Psi^{(2)}) \oplus L(\widetilde{W}_{\Psi,J}). \tag{2}$$

Put

$$J_C = L'(V_1) \oplus L'(V_2) \oplus L'(V_3) \oplus L(W_\Psi^{(2)}) \oplus L(\widetilde{W}_{\Psi,J}). \tag{3}$$

By Lemma 10,

$$L' = L(W_\Psi^{(1)}) \oplus J_C. \tag{4}$$

Note that $J_C \subseteq J$, but J_C is *not* a Lie subalgebra of J. (For example, $[y_1, y_2] \notin J_C$.) By the equation (2), we have the following $\mathbb{Z}$-module decomposition of J.

Lemma 11 $J = (L(W_\Psi^{(1)}) \cap J) \oplus J_C$.

5 L/J is torsion free

In this section we show that L/J is torsion-free $\mathbb{Z}$-module. For this we need to study the homogeneous components of J by means of Lyndon words, a filtration of the tensor powers and the symmetric powers. For the necessary material about Lyndon words and filtrations we refer the reader to ([14], [19]) and [8], respectively.

5.1 Lyndon words

In this subsection, we need some preliminaries results on Lyndon words. Let A be a totally ordered alphabet (not necessarily finite). We order the free semigroup A^+ with alphabetical order. By definition a word $w \in A^+$ is a Lyndon word if for each non-trivial factorization $w = uv$ with $u, v \in A^+$, one has $w < v$. The set of Lyndon words will be denoted by $\mathbb{L}_A$ (or briefly $\mathbb{L}$). If v is the proper right factor of maximal length of $w = uv \in \mathbb{L}$, then $u \in \mathbb{L}$, and $u < w < v$. Thus we have a recursive algorithm to construct Lyndon words. The standard factorization of each word w of length ≥ 2 is the factorization $w = u \cdot v$, where u is the smallest (proper) right factor of w for the alphabetical order. If w is a Lyndon word with standard factorization $u \cdot v$, then u, v are Lyndon words with $u < v$, $w < v$, and either u is a letter (i.e., an element of A), or the standard factorization of u is $x \cdot y$ with $y \geq v$.

Let $T(A)$ be the tensor algebra on the free $\mathbb{Z}$-module with basis A. Let $L(A)$ denote the free Lie algebra (over $\mathbb{Z}$) on A. By the Poincare-Birkhoff-Witt theorem, $T(A)$ is the universal enveloping algebra of $L(A)$ (see, for example, [11]). Namely, giving $T(A)$ the structure of Lie algebra, we may regard $L(A)$ as a Lie subalgebra of $T(A)$. We let q be the mapping of $\mathbb{L}$ into $L(A)$ defined inductively by $q(a) = a$, $a \in A$, and, for $w \in \mathbb{L} \setminus \{A\}$, $q(w) = [q(u), q(v)]$ where $w = u \cdot v$ is the standard factorization of w. For a positive integer m, we write A^m for the subset of A^+ consisting of all words of length m. The following result is well known (see, for example, [14, Proposition 5.1.4, Lemma 5.3.2]).

Lemma 12 *Let $\mathbb{L}$ be the set of Lyndon words on an alphabet A. If $u, v \in \mathbb{L}$ with $u < v$, then $uv \in \mathbb{L}$. Let $w \in \mathbb{L} \setminus A$ and its standard factorization is $u \cdot v$. Then for any $y \in \mathbb{L}$ such that $w < y$, the standard factorization of $wy \in \mathbb{L}$ is $w \cdot y$ if and only if $y \leq v$. Let $w \in \mathbb{L}^m = \mathbb{L} \cap A^m$ with $m \geq 2$. Then $q(w) = w + v$, where v belongs to the $\mathbb{Z}$-submodule of $T(A)$ spanned by those words $\widetilde{v} \in A^m$ such that $w < \widetilde{v}$.*

By Lemma 12, $q(\mathbb{L})$ is a set of linearly independent elements and so, q is injective. Furthermore, the $\mathbb{Z}$-module $L(A)$ is free with $q(\mathbb{L})$ as a basis. The elements of $q(\mathbb{L})$ are called Lyndon polynomials. We point out that the elements of $q(\mathbb{L})$ are simple Lie commutators. The proof of the following result is straightforward.

Corollary 2 *For $i = 1, \ldots, m$, let $w_i \in \mathbb{L}_A \cap A^{n_i}$. Then $q(w_1) \cdots q(w_m)$ is equal to $w_1 \cdots w_m +$ v, where v belongs to the $\mathbb{Z}$-submodule of $T(A)$ spanned by those words $\widetilde{v} \in A^{n_1+\cdots+n_m}$ such that $w_1 \cdots w_m < \widetilde{v}$.*

For $i = 1, 2, 3$, we recall that $\mathcal{V}_i = \{x_{2i-1}, x_{2i}\}$ with $x_{2i-1} < x_{2i}$, and let $\mathbb{L}_{V_i}$ denote the set of Lyndon words over $\mathcal{V}_i$. Let $T(\mathcal{V}_i) = T(V_i)$ be the tensor algebra on V_i, and consider $L(V_i)$ as a Lie subalgebra of $T(V_i)$. Furthermore, we write $q(\mathbb{L}_{V_i})$ for the $\mathbb{Z}$-basis of $L(V_i)$ corresponding to $\mathbb{L}_{V_i}$. For a positive integer m, we write $q^m(\mathbb{L}_{V_i})$ for $L^m(V_i) \cap q(\mathbb{L}_{V_i})$ which is a $\mathbb{Z}$-basis for $L^m(V_i)$. We recall that $\mathcal{W}_\Psi^{(\kappa)} = \bigcup_{c \geq 0} \mathcal{W}_{c+2,\Psi}^{(\kappa)}$ is a free generating set of $L(W_\Psi^{(\kappa)})$ (with $\kappa = 1, 2$). We arbitrarily order the elements of $\mathcal{W}_{c+2,\Psi}^{(\kappa)}$ ($\kappa = 1, 2$) for all $c \geq 0$, and extend it to $\mathcal{W}_\Psi^{(\kappa)}$ subject to $u < v$ if $u \in \mathcal{W}_{c+2,\Psi}^{(\kappa)}$ and $v \in \mathcal{W}_{e+2,\Psi}^{(\kappa)}$ with $c < e$. Let $T(\mathcal{W}_\Psi^{(\kappa)}) = T(W_\Psi^{(\kappa)})$ be the tensor algebra on $W_\Psi^{(\kappa)}$. Regard $L(W_\Psi^{(\kappa)})$ as a Lie subalgebra of $T(W_\Psi^{(\kappa)})$ and the elements of $\mathcal{W}_\Psi^{(\kappa)}$ are considered of degree 1. Write $\mathbb{L}_{W_\Psi^{(\kappa)}}$ for the set of Lyndon words over $\mathcal{W}_\Psi^{(\kappa)}$. Thus $q(\mathbb{L}_{W_\Psi^{(\kappa)}})$ is a $\mathbb{Z}$-basis for $L(W_\Psi^{(\kappa)})$. For $c \geq 0$, we write $q_{\text{grad}}^{c+2}(\mathbb{L}_{W_\Psi^{(\kappa)}}) = L_{\text{grad}}^{c+2}(W_\Psi^{(\kappa)}) \cap q(\mathbb{L}_{W_\Psi^{(\kappa)}})$.

5.2 Tensor and Symmetric Powers

Let $T = T(L^1) = T(V_1 \oplus V_2 \oplus V_3)$ be the tensor algebra on L^1. Note that $T(V_i)$ $(i = 1, 2, 3)$ is a subalgebra of $T(L^1)$. For a nonnegative integer c, let T^c denote the c-th homogeneous component of T, that is, T^c is the $\mathbb{Z}$-submodule of T spanned by all monomials $x_{i_1} \cdots x_{i_c}$ with $i_1, \ldots, i_c \in \{1, \ldots, 6\}$. Thus $T = \bigoplus_{c \geq 0} T^c$ with $T^0 = \mathbb{Z}$. As before, we consider L as a Lie subalgebra of T. An analysis of the c-th homogeneous component T^c of T, with $c \geq 1$, will help us to understand better the $(c+2)$-th homogeneous component J^{c+2} of J. For $c \geq 1$, we write $\mathrm{Part}(c)$ for the set of all partitions of c. By a composition of c we mean a sequence $\mu = (\mu_1, \ldots, \mu_\ell)$ of positive integers $\mu_1, \ldots, \mu_\ell$ satisfying $\mu_1 + \cdots + \mu_\ell = c$. If $\mu = (\mu_1, \ldots, \mu_\ell)$ is a composition of c and if we rearrange $\mu_1, \ldots, \mu_r$ in such a way to obtain a partition of c, we call it the associated partition to μ. (For example, $\mu = (1, 2, 1, 2)$ is a composition of 6 and $(2, 2, 1, 1)$ is the associated partition of μ.) We use the lexicographic order $\leq^*$ on the elements of $\mathrm{Part}(c)$. Note that the smallest partition is $(1, \ldots, 1)$ written as (1^c) and the largest partition is (c).

For a free $\mathbb{Z}$-module U of (finite) rank $r \geq 2$ with a free generating set $\{u_1, \ldots, u_r\}$, we write $S(U)$ for the free symmetric algebra on U, that is, $S(U) = \mathbb{Z}[u_1, \ldots, u_r]$. For a non-negative integer c, we write $S^c(U)$ for the c-th homogeneous component of $S(U)$ with $S^0(U) = \mathbb{Z}$. Thus $S(U) = \bigoplus_{c \geq 0} S^c(U)$. The proof of the following result is elementary.

Lemma 13 *Let $U_1, \ldots, U_\kappa$, with $\kappa \geq 2$, be free $\mathbb{Z}$-modules of finite rank. Then, for $d \geq 1$*

$$S^d(U_1 \oplus \cdots \oplus U_\kappa) \cong \bigoplus_{\substack{\nu_1 + \cdots + \nu_\kappa = d \\ \nu_1, \ldots, \nu_\kappa \geq 0}} S^{\nu_1}(U_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} S^{\nu_\kappa}(U_\kappa)$$

as $\mathbb{Z}$-modules in a natural way.

For a positive integer m, let $\mathcal{Y}_m$ be the set of all elements of T^m of the form $b_1 b_2 \cdots b_\ell$ (ℓ arbitrary) where each $b_i \in L^{\mu_i} \setminus \{0\}$ for some positive integer μ_i. Thus $\mu_i = \mathrm{deg} b_i$ for all i and $(\mu_1, \ldots, \mu_\ell)$ is a composition of m. For each partition λ of m let $\mathcal{Y}_\lambda$ denote the set of all such elements $b_1 b_2 \cdots b_\ell$ where $(\mathrm{deg} b_1, \ldots, \mathrm{deg} b_\ell)$ has λ as its associated partition. For each $\lambda \in \mathrm{Part}(m)$, let Φ_λ be the $\mathbb{Z}$-module spanned by $\mathcal{Y}_\theta$ with $\lambda \leq^* \theta$. For each λ such that $\lambda \neq (m)$ let $\lambda + 1$ be the partition of m which is next bigger than λ. Thus we have the filtration

$$T^m = \Phi_{(1^m)} \geq \cdots \geq \Phi_\lambda \geq \Phi_{\lambda+1} \geq \cdots \geq \Phi_{(m)} \geq \{0\}.$$

(Since L is free Lie algebra of rank 6, we have $\Phi_\lambda > \Phi_{\lambda+1}$.) Thus, for all λ, $\mathcal{Y}_\lambda$ spans Φ_λ modulo $\Phi_{\lambda+1}$. (We write $\Phi_{(m)+1} = \{0\}$.) Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be a composition of m with associated partition λ. For $i = 1, \ldots, \ell$, let $b_i \in L^{\mu_i} \setminus \{0\}$, and let $\pi \in \mathrm{Sym}(\ell)$. Thus $b_1 b_2 \cdots b_\ell$ and $b_{\pi(1)} b_{\pi(2)} \cdots b_{\pi(\ell)}$ belong to $\mathcal{Y}_\lambda$. As observed in [8, Lemma 3.1], we have

$$b_1 b_2 \cdots b_\ell + \Phi_{\lambda+1} = b_{\pi(1)} b_{\pi(2)} \cdots b_{\pi(\ell)} + \Phi_{\lambda+1}. \tag{5}$$

Recall that the set $\mathcal{V} = \{y_1, y_2, y_3, \psi_2(y_6), \psi_2(y_7), \psi_2(y_9), \psi_2(y_{10}), \psi_2(y_{12}), \psi_2(y_{15})\}$ is a $\mathbb{Z}$-basis of J^2. Since $J^{c+2} = [J^2, {}_cL^1]$ (by Lemma 9) and the multi-linearity of the Lie bracket, we have J^{c+2} is the $\mathbb{Z}$-module spanned by all Lie commutators of the form $[v, x_{i_1}, \ldots, x_{i_c}]$ with $i_1, \ldots, i_c \in \{1, \ldots, 6\}$. For $u = \sum * \, x_{j_1} \cdots x_{j_c} \in T^c$ with the coefficients $*$ are in $\mathbb{Z}$, we write $[v; u] = \sum * \, [v, x_{j_1}, \ldots, x_{j_c}]$. It is easily verified that for $u, w \in T^c$ and $\alpha, \beta \in \mathbb{Z}$, $[v; \alpha u + \beta w] = \alpha[v; u] + \beta[v; w]$. For each partition λ of m let $[J^2; \Phi_\lambda]$ be the $\mathbb{Z}$-module spanned

by all Lie commutators of the form $[v, b_1, \ldots, b_\ell]$ where $v \in \mathcal{V}$, $b_1 \cdots b_\ell \in \mathcal{Y}_\theta$ with $\lambda \leq^* \theta$. Note that $[J^2; \Phi_{(1^m)}] = [J^2; T^m]$, and

$$J^{m+2} = [J^2; T^m] \geq [J^2; \Phi_{(2,1^{m-1})}] \geq \cdots \geq [J^2; \Phi_{(m)}] > \{0\}.$$

The following result can be proved using the Jacobi identity in the form $[x, y, z] = [x, z, y] + [x, [y, z]]$.

Lemma 14 *Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a composition of m with associated partition λ. For $i = 1, \ldots, \ell$, let $b_i \in L^{\mu_i} \setminus \{0\}$, and let $\pi \in \mathrm{Sym}(\ell)$. Then, for $v \in J^2$, $[v, b_1, \ldots, b_\ell]$ and $[v, b_{\pi(1)}, \ldots, b_{\pi(\ell)}]$ belong to $[J^2; \Phi_\lambda]$, and*

$$[v, b_1, \ldots, b_\ell] + [J^2; \Phi_{\lambda+1}] = [v, b_{\pi(1)}, \ldots, b_{\pi(\ell)}] + [J^2; \Phi_{\lambda+1}].$$

For $\kappa = 1, 2, 3$, we similarly define $J_\kappa^{m+2} = [L^2(V_\kappa); T^m]$ and $J_\Psi^{m+2} = [W_{2,\Psi}^{(2)}; T^m]$. Since $J^2 = L^2(V_1) \oplus L^2(V_2) \oplus L^3(V_3) \oplus W_{2,\Psi}^{(2)}$, we have

$$J^{m+2} = J_1^{m+2} + J_2^{m+2} + J_3^{m+2} + J_\Psi^{m+2},$$

and, by Lemma 9,

$$J = J^2 \oplus (\bigoplus_{m \geq 1} (J_1^{m+2} + J_2^{m+2} + J_3^{m+2} + J_\Psi^{m+2})).$$

Corollary 3 *Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a composition of m with associated partition λ. For $i = 1, \ldots, \ell$, let $b_i \in L^{\mu_i} \setminus \{0\}$, and let $\pi \in \mathrm{Sym}(\ell)$. Then, for $j = 1, 2, 3$, $[y_j, b_1, \ldots, b_\ell]$ and $[y_j, b_{\pi(1)}, \ldots, b_{\pi(\ell)}]$ belong to $[L^2(V_j); \Phi_\lambda]$, and*

$$[y_j, b_1, \ldots, b_\ell] + [L^2(V_j); \Phi_{\lambda+1}] = [y_j, b_{\pi(1)}, \ldots, b_{\pi(\ell)}] + [L^2(V_j); \Phi_{\lambda+1}].$$

Furthermore, for $u \in \mathcal{W}_{2,\Psi}^{(2)}$, $[u, b_1, \ldots, b_\ell]$ and $[u, b_{\pi(1)}, \ldots, b_{\pi(\ell)}]$ belong to $[W_{2,\Psi}^{(2)}; \Phi_\lambda]$, and

$$[u, b_1, \ldots, b_\ell] + [W_{2,\Psi}^{(2)}; \Phi_{\lambda+1}] = [u, b_{\pi(1)}, \ldots, b_{\pi(\ell)}] + [W_{2,\Psi}^{(2)}; \Phi_{\lambda+1}].$$

For a positive integer n, with $n \geq 2$, we write $\widetilde{\mathcal{W}}_{n,\Psi,J}$ for the natural $\mathbb{Z}$-basis of $\widetilde{W}_{n,\Psi,J}$, and $\widetilde{\mathcal{W}}_{\Psi,J} = \bigcup_{n \geq 2} \widetilde{\mathcal{W}}_{n,\Psi,J}$. We arbitrarily order the elements of $\widetilde{\mathcal{W}}_{n,\Psi,J}$ for all $n \geq 2$, and extend it to $\widetilde{\mathcal{W}}_{\Psi,J}$ subject to $u < v$ if $u \in \widetilde{\mathcal{W}}_{n,\Psi,J}$ and $v \in \widetilde{\mathcal{W}}_{m,\Psi,J}$ with $n < m$. By Lemma 2 (for $\mathcal{A} = \widetilde{\mathcal{W}}_{\Psi,J}$), $L(\widetilde{\mathcal{W}}_{\Psi,J})'$ is freely generated by the set $\widetilde{\mathcal{W}}_{\Psi,J}^{(1)}$

$$\widetilde{\mathcal{W}}_{\Psi,J}^{(1)} = \{[a_{i_1}, \ldots, a_{i_k}] : k \geq 2, a_{i_1} > a_{i_2} \leq a_{i_3} \leq \cdots \leq a_{i_k}, a_{i_1}, \ldots, a_{i_k} \in \widetilde{\mathcal{W}}_{\Psi,J}\}.$$

We arbitrarily order the elements of $\widetilde{\mathcal{W}}_{\Psi,J}^{(1)}$ of the same degree, and elements of degree r are strictly less than the elements of degree s with $r < s$. Furthermore, for a positive integer e, let $L^{(e)}(\widetilde{\mathcal{W}}_{\Psi,J}) = (L^{(e-1)}(\widetilde{\mathcal{W}}_{\Psi,J}))'$ with $L^{(0)}(\widetilde{\mathcal{W}}_{\Psi,J}) = L(\widetilde{\mathcal{W}}_{\Psi,J})$ and $L^{(1)}(\widetilde{\mathcal{W}}_{\Psi,J}) = L(\widetilde{\mathcal{W}}_\Psi)'$. If $\widetilde{\mathcal{W}}_{\Psi,J}^{(e)}$ is an ordered free generating set for $L^{(e)}(\widetilde{\mathcal{W}}_{\Psi,J})$, then $\widetilde{\mathcal{W}}_{\Psi,J}^{(e+1)}$ is a free generating set for $L^{(e+1)}(\widetilde{\mathcal{W}}_{\Psi,J})$ with

$$\widetilde{\mathcal{W}}_{\Psi,J}^{(e+1)} = \{[a_{i_1}^{(e)}, \ldots, a_{i_k}^{(e)}] : k \geq 2, a_{i_1}^{(e)} > a_{i_2}^{(e)} \leq a_{i_3}^{(e)} \leq \cdots \leq a_{i_k}^{(e)}, a_{i_1}^{(e)}, \ldots, a_{i_k}^{(e)} \in \widetilde{\mathcal{W}}_{\Psi,J}^{(e)}\}.$$

We write

$$\widetilde{\mathcal{W}}^{\text{ext}}_{\Psi,J} = \bigcup_{d\geq 0} \widetilde{\mathcal{W}}^{(e)}_{\Psi,J}.$$

Note that the set

$$\mathcal{W}^{\star} = (\bigcup_{i=1}^{3} q(\mathbb{L}_{\mathcal{V}_i})) \cup (\bigcup_{\kappa=1}^{2} q(\mathbb{L}_{W_{\Psi}^{(\kappa)}})) \cup \widetilde{\mathcal{W}}^{\text{ext}}_{\Psi,J}$$

is a $\mathbb{Z}$-basis of L. For a non-negative integer e, let $\widetilde{\mathcal{W}}^{\text{ext},e}_{c+2,\Psi,J} = L^{c+2} \cap \widetilde{\mathcal{W}}^{(e)}_{\Psi,J}$. That is, $\widetilde{\mathcal{W}}^{\text{ext},e}_{c+2,\Psi,J}$ consists of all Lie commutators in $\widetilde{\mathcal{W}}^{(e)}_{\Psi,J}$ of total degree $c+2$. There exists a (unique) positive integer $n(c+2)$ such that the set $\widetilde{\mathcal{W}}^{\text{ext}}_{c+2,\Psi,J} = \bigcup_{e=0}^{n(c+2)} \widetilde{\mathcal{W}}^{\text{ext},e}_{c+2,\Psi,J}$ is a $\mathbb{Z}$-basis of $L^{c+2}_{\text{grad}}(\widetilde{W}_{\Psi,J})$. It is clear that $L^{c+2}_{\text{grad}}(\widetilde{W}_{\Psi,J}) = \bigoplus_{e=0}^{n(c+2)} \widetilde{W}^{\text{ext},e}_{c+2,\Psi}$ where $\widetilde{W}^{\text{ext},e}_{c+2,\Psi}$ is the $\mathbb{Z}$-span of $\widetilde{\mathcal{W}}^{\text{ext},e}_{c+2,\Psi}$. Thus, for $c \geq 0$, the set

$$\mathcal{W}^{c+2} = (\bigcup_{i=1}^{3} q^{c+2}(\mathbb{L}_{\mathcal{V}_i})) \cup (\bigcup_{\kappa=1}^{2} q^{c+2}_{\text{grad}}(\mathbb{L}_{W_{\Psi}^{(\kappa)}})) \cup \mathcal{W}^{\text{ext}}_{c+2,\Psi,J}$$

is a $\mathbb{Z}$-basis of L^{c+2} and so,

$$L^{c+2} = (\bigoplus_{i=1}^{3} L^{c+2}(V_i)) \oplus (\bigoplus_{\kappa=1}^{2} L^{c+2}_{\text{grad}}(W_{\Psi}^{(\kappa)})) \oplus L^{c+2}_{\text{grad}}(\widetilde{W}_{\Psi,J}).$$

The elements of $\mathcal{W}^{c+2}$ are ordered in such a way that the elements in $q^{c+2}(\mathbb{L}_{\mathcal{V}_i})$ $(i = 1,2,3)$, the elements in $q^{c+2}_{\text{grad}}(\mathbb{L}_{W_{\Psi}^{(\kappa)}})$ $(\kappa = 1,2)$ and the elements in $\mathcal{W}^{\text{ext}}_{c+2,\Psi,J}$ are preserve their orderings, and $u_1 \prec u_2 \prec \cdots \prec u_6$ for all $u_i \in q^{c+2}(\mathbb{L}_{\mathcal{V}_i}))$ $(i = 1,2,3)$, $u_t \in q^{c+2}_{\text{grad}}(\mathbb{L}_{W_{\Psi}^{(t-3)}}))$ $(t = 4,5)$ and $u_6 \in \mathcal{W}^{\text{ext}}_{c+2,\Psi,J}$. Choose a total ordering $\preceq$ of $\mathcal{W}^{\star} = \mathcal{X} \cup (\bigcup_{c\geq 0} \mathcal{W}^{c+2})$ in such a way all elements of $\mathcal{W}^i$ are smaller that all elements of $\mathcal{W}^j$ whenever $i < j$ with $\mathcal{W}^1 = \mathcal{X}$. (Note that the elements of each set $\mathcal{W}^i$ are already ordered.) By the Poincare-Birkhoff-Witt Theorem (see [4]), T has a $\mathbb{Z}$-basis $\mathcal{T}$ consisting of all elements of the form

$$a_1 a_2 \cdots a_\ell \quad (\ell \geq 0, a_1, \ldots, a_\ell \in \mathcal{W}^{\star}, a_1 \preceq \cdots \preceq a_\ell).$$

The above elements of $\mathcal{T}$ are distinct as written. For $c \geq 1$, we write $\mathcal{T}_c$ for the set of the above elements of degree c. Clearly, $\mathcal{T}_c$ is a $\mathbb{Z}$-basis of T^c. Also, for $\lambda \in \text{Part}(c)$, we write $\mathcal{T}_\lambda$ for the set of elements of the above form in $\mathcal{T}_c$ such that the composition $(\text{deg}a_1, \ldots, \text{deg}a_\ell)$ has λ as associated partition. As pointed out in [8, at the bottom of p. 183] the elements of $\mathcal{T}_\lambda$ taken modulo $\Phi_{\lambda+1}$ form a $\mathbb{Z}$-basis of $\Phi_\lambda/\Phi_{\lambda+1}$. Any partition $\lambda \in \text{Part}(c)$ is written as $\lambda = (c^{n(c)}, (c-1)^{n(c-1)}, \ldots, 1^{n(1)})$ with $n(c), \ldots, n(1)$ non-negative integers. For such a partition λ, we define

$$L^\lambda = S^{n(1)}(L^1) \otimes_{\mathbb{Z}} S^{n(2)}(L^2) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} S^{n(c)}(L^c).$$

As observed in [8, pp. 183-184], $\Phi_\lambda/\Phi_{\lambda+1} \cong L^\lambda$ as $\mathbb{Z}$-modules, and the basis $\{w + \Phi_{\lambda+1} : w \in \mathcal{T}_\lambda\}$ of $\Phi_\lambda/\Phi_{\lambda+1}$ consists of all elements of the form

$$a_1^{(1)} \cdots a_1^{(n(1))} a_2^{(1)} \cdots a_2^{(n(2))} \cdots a_c^{(1)} \cdots a_c^{(n(c))} + \Phi_{\lambda+1} \tag{6}$$

where $a_1^{(1)} \preceq \cdots \preceq a_1^{(n(1))} \preceq a_2^{(1)} \preceq \cdots \preceq a_2^{(n(2))} \preceq \cdots \preceq a_c^{(1)} \preceq \cdots \preceq a_c^{(n(c))}$ and $a_i^{(1)}, \cdots, a_i^{(n(i))} \in \mathcal{W}^i$ for $i = 1, \ldots, c$. (For $i = 1$, $\mathcal{W}^1 = \mathcal{X}$.)

5.3 An analysis of L^λ

We define $\mathcal{G}_\lambda = \mathcal{G}_{1,n(1)} \times \cdots \times \mathcal{G}_{c,n(c)}$ and $\mathcal{B}_\lambda = \mathcal{B}_{1,n(1)} \times \cdots \times \mathcal{B}_{c,n(c)}$, where

$$\mathcal{G}_{1,n(1)} = \{(\nu_{1,1}, \nu_{2,1}, \nu_{3,1}) \in \mathbb{N}_0^3 : \nu_{1,1} + \nu_{2,1} + \nu_{3,1} = n(1)\},$$

for $\kappa = 2, 3$,

$$\mathcal{G}_{\kappa,n(\kappa)} = \{(\nu_{1,\kappa}, \ldots, \nu_{5,\kappa}) \in \mathbb{N}_0^5 : \nu_{1,\kappa} + \cdots + \nu_{5,\kappa} = n(\kappa)\},$$

and, for $4 \leq \kappa \leq c$,

$$\mathcal{G}_{\kappa,n(\kappa)} = \{(\nu_{1,\kappa}, \ldots, \nu_{6,\kappa}) \in \mathbb{N}_0^6 : \nu_{1,\kappa} + \cdots + \nu_{6,\kappa} = n(\kappa)\},$$

($\mathbb{N}_0$ denotes the set of all non-negative integers.)

$$\mathcal{B}_{1,n(1)} = \{(S^{\nu_{1,1}}(V_1), S^{\nu_{2,1}}(V_2), S^{\nu_{3,1}}(V_3)) : (\nu_{1,1}, \nu_{2,1}, \nu_{3,1}) \in \mathcal{G}_{1,n(1)}\},$$

for $\kappa = 2, 3$,

$$\mathcal{B}_{\kappa,n(\kappa)} = \{(S^{\nu_{1,\kappa}}(L^\kappa(V_1)), \ldots, S^{\nu_{5,\kappa}}(L^\kappa(W_\Psi^{(2)})) : (\nu_{1,\kappa}, \ldots, \nu_{5,\kappa}) \in \mathcal{G}_{\kappa,n(\kappa)}\},$$

and, for $4 \leq \kappa \leq c$,

$$\mathcal{B}_{\kappa,n(\kappa)} = \{(S^{\nu_{1,\kappa}}(L^\kappa(V_1)), \ldots, S^{\nu_{6,\kappa}}(L^\kappa_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})) : (\nu_{1,\kappa}, \ldots, \nu_{6,\kappa}) \in \mathcal{G}_{\kappa,n(\kappa)}\}.$$

By applying Lemma 13 (using the above $\mathbb{Z}$-module decompositions of $L^1, \ldots, L^c$, respectively), by the associativity of $\otimes_\mathbb{Z}$ and by the distributivity of $\oplus_\mathbb{Z}$ with $\otimes_\mathbb{Z}$, we have

$$L^\lambda \cong \bigoplus_{(X_1,\ldots,X_c) \in \mathcal{B}_\lambda} (\overline{X}_1 \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \overline{X}_c).$$

(For example, if $X_1 = (S^{\nu_{1,1}}(V_1), S^{\nu_{2,1}}(V_2), S^{\nu_{3,1}}(V_3)) \in \mathcal{B}_{1,n(1)}, \ldots, X_c = (S^{\nu_{1,c}}(L^c(V_1)), \ldots, S^{\nu_{6,c}}(L^c_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})) \in \mathcal{B}_{c,n(c)}$, by $\overline{X}_1 \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \overline{X}_c$, we mean the expression

$$S^{\nu_{1,1}}(V_1) \otimes_\mathbb{Z} S^{\nu_{2,1}}(V_2) \otimes_\mathbb{Z} S^{\nu_{3,1}}(V_3) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} S^{\nu_{1,c}}(L^c(V_1)) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} S^{\nu_{6,c}}(L^c_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})).$$

The elements of $\mathcal{G}_{\kappa,n(\kappa)}$ are denoted by $Y_{1,n(1),3}$ (for $\kappa = 1$), $Y_{\kappa,n(\kappa),5}$ (for $\kappa = 2, 3$) and $Y_{\kappa,n(\kappa),6}$ (for $4 \leq \kappa \leq c$). For $Y = (Y_{1,n(1),3}, \ldots, Y_{c,n(c),6}) \in \mathcal{G}_\lambda$, we correspond a unique element of the basis of $\Phi_\lambda/\Phi_{\lambda+1}$ of the form (6) where the first $\nu_{1,1}$ elements are in $\mathcal{V}_1$ (if $\nu_{1,1} \geq 1$), the second $\nu_{2,1}$ elements are in $\mathcal{V}_2$ (if $\nu_{2,1} \geq 1$), the next $\nu_{3,1}$ elements are in $\mathcal{V}_3$ (if $\nu_{3,1} \geq 1$) with $(\nu_{1,1}, \nu_{2,1}, \nu_{3,1}) \in Y_{1,n(1),3}$ and so on, and vice versa. Furthermore, for such $Y \in \mathcal{G}_\lambda$, we associate the nonnegative integers $m_{\lambda,Y}(1), \ldots, m_{\lambda,Y}(6)$ defined as follows:

$$\begin{array}{rcl}
m_{\lambda,Y}(t) & = & \nu_{t,1} \cdot 1 + \nu_{t,2} \cdot 2 + \cdots + \nu_{t,c} \cdot c \quad (t = 1, 2, 3), \\
m_{\lambda,Y}(t) & = & \nu_{t,2} \cdot 2 + \cdots + \nu_{t,c} \cdot c \quad (t = 4, 5) \\
\text{and} & & \\
m_{\lambda,Y}(6) & = & \nu_{6,4} \cdot 4 + \cdots + \nu_{6,c} \cdot c.
\end{array}$$

Note that $m_{\lambda,Y}(1) + \cdots + m_{\lambda,Y}(6) = c$ for all $Y \in \mathcal{G}_\lambda$. By the equation (5), we rearrange the above elements in such a way the first $m_{\lambda,Y}(1)$ elements are in $q(\mathbb{L}_{V_1})$, the second $m_{\lambda,Y}(2)$ elements are in $q(\mathbb{L}_{V_2})$ and so on. More precisely, for $Y = (Y_{1,n(1),3}, \ldots, Y_{c,n(c),6}) \in \mathcal{G}_\lambda$ we have the

partitions $p_{m_{\lambda,Y}(t)}$ of $m_{\lambda,Y}(t)$, $t=1,\ldots,6$, where $p_{m_{\lambda,Y}(t)} = (c^{\nu_{t,c}}, (c-1)^{\nu_{t,c-1}}, \ldots, 2^{\nu_{t,2}}, 1^{\nu_{t,1}})$ (for $t=1,2,3$), $p_{m_{\lambda,Y}(t)} = (c^{\nu_{t,c}}, (c-1)^{\nu_{t,c-1}}, \ldots, 2^{\nu_{t,2}})$ (for $t=4,5$) and $p_{m_{\lambda,Y}(6)} = (c^{\nu_{6,c}}, (c-1)^{\nu_{6,c-1}}, \ldots, 4^{\nu_{6,c}})$. It is clearly enough that for $\lambda = (c^{n(c)}, (c-1)^{n(c-1)}, \ldots, 1^{n(1)}) \in \mathrm{Part}(c)$ with $n(c), \ldots, n(1)$ nonnegative integers

$$L^{\lambda} \cong \bigoplus_{Y \in \mathcal{G}_{\lambda}} (L^{p_{m_{\lambda,Y}(1)}}(V_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L^{p_{m_{\lambda,Y}(6)}}(\widetilde{W}_{\Psi,J})), \tag{7}$$

where $L^{p_{m_{\lambda,Y}(t)}}(V_t) = S^{\nu_{t,1}}(V_t) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} S^{\nu_{t,c}}(L^c(V_t))$ $(t=1,2,3)$, $L^{p_{m_{\lambda,Y}(t)}}(W_{\Psi}^{(t-3)}) = S^{\nu_{t,2}}(L^2_{\mathrm{grad}}(W_{\Psi}^{(t-3)})) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} S^{\nu_{t,c}}(L^c_{\mathrm{grad}}(W_{\Psi}^{(t-3)}))$ $(t=4,5)$ and $L^{p_{m_{\lambda,Y}(6)}}(\widetilde{W}_{\Psi,J}) = S^{\nu_{6,4}}(L^4_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} S^{\nu_{6,c}}(L^c_{\mathrm{grad}}(\widetilde{W}_{\Psi,J}))$.

5.4 A $\mathbb{Z}$-basis of $\Phi_{\lambda}/\Phi_{\lambda+1}$

For $t=1,2,3$, we write

$$w(p_{m_{\lambda,Y}(t)}) = a_{1,t}^{(1)} \cdots a_{1,t}^{(\nu_{t,1})} a_{2,t}^{(1)} \cdots a_{2,t}^{(\nu_{t,2})} \cdots a_{c,t}^{(1)} \cdots a_{c,t}^{(\nu_{t,c})} \in T(L^1),$$

where $a_{j,t}^{(1)}, \ldots, a_{j,t}^{(\nu_{t,j})}, \in q^j(\mathbb{L}_{V_t})$, $j=1,\ldots,c$, for $t=4,5$

$$w(p_{m_{\lambda,Y}(t)}) = a_{2,t}^{(1)} \cdots a_{2,t}^{(\nu_{t,2})} \cdots a_{c,t}^{(1)} \cdots a_{c,t}^{(\nu_{t,c})} \in T(L^1),$$

where $a_{j,t}^{(1)}, \ldots, a_{j,t}^{(\nu_{t,j})} \in q^j_{\mathrm{grad}}(\mathbb{L}_{W_{\Psi}^{(t-3)}})$, $j=2,\ldots,c$, and

$$w(p_{m_{\lambda,Y}(6)}) = a_{4,6}^{(1)} \cdots a_{4,6}^{(\nu_{6,4})} \cdots a_{c,6}^{(1)} \cdots a_{c,6}^{(\nu_{6,c})} \in T(L^1),$$

where $a_{j,6}^{(1)}, \ldots, a_{j,6}^{(\nu_{6,j})} \in \mathcal{W}^{\mathrm{ext}}_{j,\Psi,J}$, $j=4,\ldots,c$. Furthermore, we write

$$w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)}) = w(p_{m_{\lambda,Y}(1)}) \cdots w(p_{m_{\lambda,Y}(6)}) \in T(L^1).$$

Lemma 15 *The set $\{w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)}) + \Phi_{\lambda+1} : Y \in \mathcal{G}_{\lambda}\}$ spans $\Phi_{\lambda}/\Phi_{\lambda+1}$. In particular, it is a $\mathbb{Z}$-basis of $\Phi_{\lambda}/\Phi_{\lambda+1}$.*

Proof. Indeed, writing $w = a_1^{(1)} \cdots a_1^{(n(1))} \cdots a_c^{(1)} \cdots a_c^{(n(c))} = b_1 \cdots b_{\ell}$, then, by a suitable permutation $\pi \in \mathrm{Sym}(\ell)$, $b_{\pi(1)} \cdots b_{\pi(\ell)} + \Phi_{\lambda+1} = w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)}) + \Phi_{\lambda+1}$ for a unique $Y \in \mathcal{G}_{\lambda}$. Since the set $\{w + \Phi_{\lambda+1} : w \in \mathcal{T}_{\lambda}\}$ is a $\mathbb{Z}$-basis, we have the required result. □

Remark 1 *We point out that, for any $\pi \in \mathrm{Sym}(6)$, we have $w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)}) + \Phi_{\lambda+1} = w(p_{m_{\lambda,Y}(\pi(1))}, \ldots, p_{m_{\lambda,Y}(\pi(6))}) + \Phi_{\lambda+1}$.*

5.5 A presentation of J^{c+2} via a filtration

For $v \in \mathcal{V}$ and $w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})$ $(Y \in \mathcal{G}_{\lambda})$ as above, we denote

$$[v; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})] = [v, w(p_{m_{\lambda,Y}(1)}), \ldots, w(p_{m_{\lambda,Y}(6)})] \in L^{c+2}.$$

We write $[J^2;\Phi_\lambda/\Phi_{\lambda+1}]$ for the $\mathbb{Z}$-submodule of $[J^2;T^c]$ spanned by all Lie commutators of the form $[v;w(p_{m_{\lambda,Y}(1)},\ldots,p_{m_{\lambda,Y}(6)})]$ with $v\in\mathcal{V}$ and $Y\in\mathcal{G}_\lambda$. Clearly, for $c\geq 0$,

$$J^{c+2}=\sum_{\lambda\in\mathrm{Part}(c)}[J^2;\Phi_\lambda/\Phi_{\lambda+1}].$$

For $\kappa=1,2,3$, we similarly define $[L^2(V_\kappa);\Phi_\lambda/\Phi_{\lambda+1}]$ and $[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}]$. Since $J^2=L^2(V_1)\oplus L^2(V_2)\oplus L^2(V_3)\oplus W^{(2)}_{2,\Psi}$, we get

$$[J^2;\Phi_\lambda/\Phi_{\lambda+1}]=[L^2(V_1);\Phi_\lambda/\Phi_{\lambda+1}]+[L^2(V_2);\Phi_\lambda/\Phi_{\lambda+1}]+[L^2(V_3);\Phi_\lambda/\Phi_{\lambda+1}]+[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}].$$

Writing

$$J^{c+2}_\kappa=\sum_{\lambda\in\mathrm{Part}(c)}[L^2(V_\kappa);\Phi_\lambda/\Phi_{\lambda+1}]\ \ (\kappa=1,2,3)\ \ \text{and}\ \ J^{c+2}_\Psi=\sum_{\lambda\in\mathrm{Part}(c)}[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}],$$

we have

$$J^{c+2}=J^{c+2}_1+J^{c+2}_2+J^{c+2}_3+J^{c+2}_\Psi.$$

For $\lambda\in\mathrm{Part}(c)$, and $\kappa=1,2,3$, we let

$$J^{c+2}_{\kappa,\lambda}=\sum_{\substack{\theta\in\mathrm{Part}(c)\\ \lambda\leq^*\theta}}[L^2(V_\kappa);\Phi_\lambda/\Phi_{\lambda+1}]\ \ \text{and}\ \ J^{c+2}_{\Psi,\lambda}=\sum_{\substack{\theta\in\mathrm{Part}(c)\\ \lambda\leq^*\theta}}[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}].$$

Note that $J^{c+2}_{\kappa,\lambda}=[L^2(V_\kappa);\Phi_\lambda/\Phi_{\lambda+1}]+J^{c+2}_{\kappa,\lambda+1}$ and $J^{c+2}_{\Psi,\lambda}=[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}]+J^{c+2}_{\Psi,\lambda+1}$. Furthermore, $J^{c+2}_\kappa=J^{c+2}_{\kappa,(1^c)}$ $(\kappa=1,2,3)$ and $J^{c+2}_\Psi=J^{c+2}_{\Psi,(1^c)}$ and so,

$$J^{c+2}=J^{c+2}_{1,(1^c)}+J^{c+2}_{2,(1^c)}+J^{c+2}_{3,(1^c)}+J^{c+2}_{\Psi,(1^c)}.$$

5.6 A decomposition of L^2, L^3

In this section we give a $\mathbb{Z}$-module decomposition of L^2 and L^3.

Lemma 16 *We have the following $\mathbb{Z}$-module decompositions*
(I) $L^2=W^{(1)}_{2,\Psi}\oplus J^2$. *In particular, L^2/J^2 is torsion-free.*
(II) $J^3=(\bigoplus_{i=2}^3[L^2(V_1),V_i])\oplus(\bigoplus_{\substack{i=1\\ i\neq 2}}^3[L^2(V_2),V_i])\oplus(\bigoplus_{i=1}^2[L^2(V_3),V_i])\oplus[[V_3,V_2]^{(2)},V_1]\oplus J^3_C$.
(III) $L^3=(W^{(1)}_{3,\Psi})^*\oplus J^3$, *where $(W^{(1)}_{3,\Psi})^*$ denotes the $\mathbb{Z}$-submodule of $W^{(1)}_{3,\Psi}$ spanned by the set* $(\mathcal{W}^{(1)}_{3,\Psi})^*=\{[y_4,x_i],[y_4,x_5],[y_5,x_3],[y_5,x_5],[y_5,x_6],[y_8,x_3],[y_8,x_5],[y_{11},x_2],[y_{11},x_4],[y_{11},x_6],[y_{13},x_5],[y_{14},x_3],[y_{14},x_4],[y_{14},x_5],i=1,2,3\}$. *In particular, L^3/J^3 is torsion-free.*

Proof. (I) This is straightforward.

(II) Note that $J^3_\kappa=[L^2(V_\kappa),L^1]$ $(\kappa=1,2,3)$ and $J^3_\Psi=[W^{(2)}_{2,\Psi},L^1]$. Since $L^1=V_1\oplus V_2\oplus V_3$, $[W^{(2)}_{2,\Psi},L^1]=[[V_3,V_2]^{(2)},V_1]+W^{(2)}_{3,\Psi}$ and $J^3=J^3_1+J^3_2+J^3_3+J^3_\Psi$, we have

$$J^3=(\sum_{i=2}^3[L^2(V_1),V_i])+(\sum_{\substack{i=1\\ i\neq 2}}^3[L^2(V_2),V_i])+(\sum_{i=1}^2[L^2(V_3),V_i])+[[V_3,V_2]^{(2)},V_1]+J^3_C.$$

Using the Jacobi identity and the definition of ψ_2, we have

$$\begin{array}{rcl}
[y_1, x_3] &=& -[y_5, x_1] + [y_4, x_2] - [\psi_2(y_6), x_1] + [\psi_2(y_7), x_1], \\
[y_1, x_4] &=& [y_5, x_1] + [y_5, x_2] - [\psi_2(y_7), x_1], \\
[y_1, x_5] &=& -[\psi_2(y_{10}), x_1] + [y_8, x_1] - [y_8, x_2], \\
[y_1, x_6] &=& -[y_{11}, x_1] - [y_8, x_2] - [\psi_2(y_9), x_2], \qquad (C1)
\end{array}$$

$$\begin{array}{rcl}
[y_2, x_1] &=& [y_5, x_3] - [y_4, x_4], \\
[y_2, x_2] &=& -[y_5, x_3] - [y_5, x_4] + [\psi_2(y_7), x_3] - [\psi_2(y_6), x_4] + [\psi_2(y_7), x_4], \\
[y_2, x_5] &=& -[y_{14}, x_3] + [\psi_2(y_{12}), x_4] - [y_{13}, x_4], \\
[y_2, x_6] &=& [y_{13}, x_4] - [\psi_2(y_{15}), x_3] + [y_{13}, x_3], \qquad (C2)
\end{array}$$

$$\begin{array}{rcl}
[y_3, x_1] &=& [\psi_2(y_9), x_5] - [y_8, x_5] - [y_8, x_6], \\
[y_3, x_2] &=& -[\psi_2(y_{10}), x_6] + [y_{11}, x_5] + [y_8, x_6], \\
[y_3, x_3] &=& [y_{13}, x_5] - [\psi_2(y_{12}), x_6] + [y_{13}, x_6], \\
[y_3, x_4] &=& [\psi_2(y_{15}), x_5] - [y_{13}, x_5] - [y_{14}, x_6], \qquad (C3)
\end{array}$$

$$\begin{array}{rcl}
[\psi_2(y_{12}), x_1] &=& [\psi_2(y_9), x_3] - [y_4, x_5] - [y_8, x_3] - [y_4, x_6], \\
[\psi_2(y_{12}), x_2] &=& [\psi_2(y_{10}), x_3] - [y_8, x_3] - [y_5, x_5] - [\psi_2(y_6), x_5] + [\psi_2(y_7), x_5] + \\
&& [y_{11}, x_3] - [\psi_2(y_6), x_6] + [\psi_2(y_7), x_6] - [y_5, x_6], \qquad (C4)
\end{array}$$

$$[\psi_2(y_{15}), x_1] = [\psi_2(y_9), x_4] - [y_8, x_4] - [y_5, x_6] + [\psi_2(y_9), x_3] - [y_8, x_3] - [y_4, x_6],$$

and

$$[\psi_2(y_{15}), x_2] = [y_{11}, x_4] - [\psi_2(y_6), x_6] + [y_{11}, x_3] \qquad (C5).$$

Write $\widetilde{\mathcal{W}_3}$ for the set of the above elements. Working in $W_{3,\Psi}$, by direct calculations, we show that the elements of $\widetilde{\mathcal{W}_3}$ are $\mathbb{Z}$-linear independent. Hence, the sum

$$(\sum_{i=2}^{3}[L^2(V_1), V_i]) + (\sum_{\substack{i=1\\i\neq 2}}^{3}[L^2(V_2), V_i]) + (\sum_{i=1}^{2}[L^2(V_3), V_i]) + [[V_3, V_2]^{(2)}, V_1]$$

is direct, and $\widetilde{\mathcal{W}_3}$ is a $\mathbb{Z}$-basis of the aforementioned direct sum.

(III) Since $J^3 = \langle\widetilde{\mathcal{W}_3}\rangle + J^3_C$ and $\langle\widetilde{\mathcal{W}_3}\rangle \cap J^3_C = \{0\}$, we have $J^3 = \langle\widetilde{\mathcal{W}_3}\rangle \oplus J^3_C$. Next, we claim that

$$J^3 = (J \cap W^{(1)}_{3,\Psi}) \oplus J^3_C.$$

It is enough to show that

$$\langle\widetilde{\mathcal{W}_3}\rangle \subseteq (J \cap W^{(1)}_{3,\Psi}) \oplus J^3_C$$

which follows from the equations $(C1)-(C5)$. From the above equations $(C1)-(C5)$, and since $J\cap W^{(1)}_{3,\Psi} \cong \langle\widetilde{\mathcal{W}_3}\rangle$ as $\mathbb{Z}$-modules, we may easily describe the $\mathbb{Z}$-module $J\cap W^{(1)}_{3,\Psi}$. Let $(W^{(1)}_{3,\Psi})^*$ be the $\mathbb{Z}$-submodule of $W^{(1)}_{3,\Psi}$ spanned by the set $(\mathcal{W}^{(1)}_{3,\Psi})^* = \{[y_4, x_i], [y_4, x_5], [y_5, x_3], [y_5, x_5], [y_5, x_6], [y_8, x_3], [y_8, x_5], [y_{11}, x_2], [y_{11}, x_4], [y_{11}, x_6], [y_{13}, x_5], [y_{14}, x_3], [y_{14}, x_4], [y_{14}, x_5], i = 1, 2, 3\}$. It is easily verified that

$$W^{(1)}_{3,\Psi} = (J \cap W^{(1)}_{3,\Psi}) \oplus (W^{(1)}_{3,\Psi})^*.$$

Thus

$$L^3 = (W^{(1)}_{3,\Psi})^* \oplus J^3$$

that is, the required result. $\square$

5.7 Three technical results

In this section we shall show three technical results which are using in the proof of our main result Theorem 2. Let $u \in q^{c+2}_{\mathrm{grad}}(\mathbb{L}_{W^{(\kappa)}_{\Psi}})$. Then there exists a unique $w \in \mathbb{L}_{W^{(\kappa)}_{\Psi}}$ such that $u = q(w)$ and w has length $c+2$ in $x_1, \ldots, x_6$. Since $w \in \mathbb{L}_{W^{(\kappa)}_{\Psi}}$, w has a "degree "in terms of elements of $\mathcal{W}^{(\kappa)}_{\Psi}$ as well. To express these information, we write $w^{(\kappa)}_{[s,t]}$ for a Lyndon word $w \in (\mathcal{W}^{(\kappa)}_{\Psi})^{+}$ of degree s (in terms of $x_1, \ldots, x_6$) and degree t (in terms of the elements of $\mathcal{W}^{(\kappa)}_{\Psi}$). (For example, $y_4, y_{11} \in \mathcal{W}^{(1)}_{\Psi}$ are Lyndon words of type $w^{(1)}_{[2,1]}$, and $[\psi_2(y_{10}), \psi_2(y_{12})]$ is a Lyndon word over $\mathcal{W}^{(2)}_{\Psi}$ of type $w^{(2)}_{[4,2]}$.) For positive integers s and t, with $s \geq 2$, let $W^{(\kappa),t}_{s,\Psi}$ be the $\mathbb{Z}$-submodule of $L^{s}_{\mathrm{grad}}(W^{(\kappa)}_{\Psi})$ spanned by all Lyndon polynomials $\zeta^{(\kappa)}_{[s,t]} = q^{s}_{\mathrm{grad}}(w^{(\kappa)}_{[s,t]})$. By the definition of $L^{s}_{\mathrm{grad}}(W^{(\kappa)}_{\Psi})$, there exists a (unique) positive integer d_s such that

$$L^{s}_{\mathrm{grad}}(W^{(\kappa)}_{\Psi}) = \bigoplus_{t=1}^{d_s} W^{(\kappa),t}_{s,\Psi}.$$

Furthermore, we write for $e \in \{1, \ldots, d_s\}$

$$L^{s}_{\mathrm{grad,e}}(W^{(\kappa)}_{\Psi}) = \bigoplus_{t=e}^{d_s} W^{(\kappa),t}_{s,\Psi}.$$

Note that $L^{s}_{\mathrm{grad},1}(W^{(\kappa)}_{\Psi}) = L^{s}_{\mathrm{grad}}(W^{(\kappa)}_{\Psi})$.

Lemma 17 *Let r, s be positive integers, with $r, s \geq 2$, and $\kappa, \mu \in \{1, 2\}$. Let b be a Lyndon polynomial in $L^{r}(V_{\mu})$. Then, for $t \in \{1, \ldots, d_s\}$, $[\zeta^{(\kappa)}_{[s,t]}, b] = \zeta^{(\kappa)}_{[s+r,t]} + w'_{s+r,t}$ with $w^{(\kappa)}_{[s,t]} < w^{(\kappa)}_{[s+r,t]}$ and $w'_{s+r,t} \in L^{s+r}_{\mathrm{grad}}(W_{\Psi})$. Furthermore, $w^{(\kappa)}_{[s+r,t]}$ is the smallest word in the expression $\zeta^{(\kappa)}_{[s+r,t]} + w'_{s+r,t}$.*

Proof. We shall show our claim for $\mu = 2$ and $\kappa = 1$. Similar arguments may be applied to the other cases. Let $s \geq 2$. We induct on t. For $t = 1$, $W^{(1),1}_{s,\Psi} = W^{(1)}_{s,\Psi}$. Thus $\zeta^{(1)}_{[s,1]} = q^{s}_{\mathrm{grad}}(w^{(1)}_{[s,1]}) = w^{(1)}_{[s,1]}$ $(\in \mathcal{W}^{(1)}_{s,\Psi})$ has the form $v^{(\mu,1,\Psi)}_{s,(\alpha,\beta,\gamma)}$, $v^{(6,\nu,\Psi)}_{s,(\alpha,\beta,\gamma)}$, or $v^{(5,4,\Psi)}_{s,(\alpha,\beta,\gamma)}$ with $\alpha+\beta+\gamma = s-2$, $\mu \in \{3,4,5\}$ and $\nu \in \{2,3\}$. Let $b = q^{r}(u)$, $u \in \mathbb{L}^{r}_{V_2}$, a basis element of $L^{r}(V_2)$. By Lemma 12, $b = u + v$, where v belongs to the $\mathbb{Z}$-submodule of $T(V_2)$ spanned by $\widetilde{v} \in \mathcal{V}^{r}_{2}$ and $u < \widetilde{v}$. Thus

$$[\zeta^{(1)}_{[s,1]}, b] = [\zeta^{(1)}_{[s,1]}, x_3, \ldots, x_4] + \sum * \, [\zeta^{(1)}_{[s,1]}, x_{i_1}, \ldots, x_{i_r}], \tag{†}$$

where $u = x_3 \cdots x_4$, $\widetilde{v} = x_{i_1} \cdots x_{i_r}$, $i_1, \ldots, i_r \in \{3,4\}$ and the coefficients $*$ are in $\mathbb{Z}$. Using the Jacobi identity in the form $[x,y,z] = [x,z,y] + [x,[y,z]]$, by the equations (D) and the definition of ψ_2, we have

$$[\zeta^{(1)}_{[s,1]}, x_3, \ldots, x_4] = \zeta^{(1)}_{[s+r,1]} + w_{s+r,1}$$

where $\zeta^{(1)}_{[s+r,1]}$ has the form $v^{(\mu,1,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, $v^{(6,\nu,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, or $v^{(5,4,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, $w_{s+r,1} \in L^{s+r}_{\mathrm{grad,e}}(W^{(1)}_{\Psi}) \oplus L^{s+r}_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})$ for some $e \geq 2$. Thus $\zeta^{(1)}_{[s+r,1]}$ does not occur in the expression of $w_{s+r,1}$. Note

that, by the ordering on $\mathcal{W}_\Psi$ (section 3.3), $w^{(1)}_{[s,1]} < w^{(1)}_{[s+r,1]}$. We apply similar arguments to each $[\zeta^{(1)}_{[s,1]}, x_{i_1}, \ldots, x_{i_r}]$ appearing in the right hand side of the equation (†). More precisely,

$$[\zeta^{(1)}_{[s,1]}, x_{i_1}, \ldots, x_{i_r}] = \zeta^{(1),i_1,\ldots,i_r}_{[s+r,1]} + w_{s+r,1,i_1,\ldots,i_r}$$

where $\zeta^{(1),i_1,\ldots,i_r}_{[s+r,1]}$ has the form $v^{(\mu,1,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, $v^{(6,\nu,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, or $v^{(5,4,\Psi)}_{s+r,(\alpha,\beta+r,\gamma)}$, $w_{s+r,1,i_1,\ldots,i_r} \in L^{s+r}_{\text{grad,e}}(W^{(1)}_\Psi) \oplus L^{s+r}_{\text{grad}}(\widetilde{W}_{\Psi,J})$ for some $e \geq 2$. Since $u < \widetilde{v}$, we obtain by the ordering on $\mathcal{W}_\Psi$, $w^{(1)}_{[s+r,1]} < w^{(1)}_{[s+r,1,i_1,\ldots,i_r]}$ and so, $w^{(1)}_{[s+r,1]}$ is the smallest Lyndon word occurring in the right hand side of the equation (†). Therefore, for all $s, r \geq 2$,

$$[\zeta^{(1)}_{[s,1]}, b] = \zeta^{(1)}_{[s+r,1]} + w'_{s+r,1} \quad \text{with} \quad w^{(1)}_{[s,1]} < w^{(1)}_{[s+r,1]}, \tag{‡}$$

where $w'_{s+r,1} \in L^{s+r}_{\text{grad}}(W^{(1)}_\Psi) \oplus L^{s+r}_{\text{grad}}(\widetilde{W}_{\Psi,J})$. Furthermore, $w^{(1)}_{[s+r,1]}$ is the smallest word occurring in the right hand side of the equation (‡).

We assume that for $t \in \{1, \ldots, d_s\}$, our claim is valid for all $t' < t$, all $s_1, s_2 < s$ and all $r \geq 2$. We shall show our claim for t. Let $\zeta^{(1)}_{[s,t]} = q^s_{\text{grad}}(w^{(1)}_{[s,t]})$, and let $w^{(1)}_{[s_1,t_1]} \cdot w^{(1)}_{[s_2,t_2]}$ ($s = s_1 + s_2$ and $t = t_1 + t_2$) be the standard factorization of $w^{(1)}_{[s,t]}$. Then $w^{(1)}_{[s_1,t_1]}, w^{(1)}_{[s_2,t_2]}$ are Lyndon words with $w^{(1)}_{[s_1,t_1]} < w^{(1)}_{[s_2,t_2]}$ and $w^{(1)}_{[s,t]} < w^{(1)}_{[s_2,t_2]}$. Then

$$\begin{array}{rcl} [\zeta^{(1)}_{[s,t]}, b] & = & [q^s_{\text{grad}}(w^{(1)}_{[s,t]}), b] \\ & = & [[\zeta^{(1)}_{[s_1,t_1]}, \zeta^{(1)}_{[s_2,t_2]}], b] \\ (\text{Jacobi identity}) & = & [[\zeta^{(1)}_{[s_1,t_1]}, b], \zeta^{(1)}_{[s_2,t_2]}] - [[\zeta^{(1)}_{[s_2,t_2]}, b], \zeta^{(1)}_{[s_1,t_1]}]. \end{array}$$

By our inductive hypothesis, for $j = 1, 2$,

$$[\zeta^{(1)}_{[s_j,t_j]}, b] = \zeta^{(1)}_{[s_j+r,t_j]} + w'_{s_j+r,t_j} \text{ with } w^{(1)}_{[s_j,t_j]} < w^{(1)}_{[s_j+r,t_j]},$$

$w'_{s_j+r,t_j} \in L^{s_j+r}_{\text{grad}}(W_\Psi)$ and $w^{(1)}_{[s_j+r,t_j]}$ is the smallest word in the expression of $\zeta^{(1)}_{[s_j+r,t_j]} + w'_{s_j+r,t_j}$. Therefore

$$[[\zeta^{(1)}_{[s_1,t_1]}, b], \zeta^{(1)}_{[s_2,t_2]}] = [\zeta^{(1)}_{[s_1+r,t_1]}, \zeta^{(1)}_{[s_2,t_2]}] + [w'_{s_1+r,t_1}, \zeta^{(1)}_{[s_2,t_2]}]$$

and

$$[[\zeta^{(1)}_{[s_2,t_2]}, b], \zeta^{(1)}_{[s_1,t_1]}] = [\zeta^{(1)}_{[s_2+r,t_2]}, \zeta^{(1)}_{[s_1,t_1]}] + [w'_{s_2+r,t_2}, \zeta^{(1)}_{[s_1,t_1]}].$$

Since $[u, v] = uv - vu$ for all $u, v \in T(L^1)$, and by using Lemma 12, we have

$$[\zeta^{(1)}_{[s_1,t_1]}, \zeta^{(1)}_{[s_2+r,t_2]}] = w^{(1)}_{[s_1,t_1]} w^{(1)}_{[s_2+r,t_2]} + \sum * \, \widetilde{w} \tag{♭}$$

where $w^{(1)}_{[s_1,t_1]} w^{(1)}_{[s_2+r,t_2]} \in \mathbb{L}^{s+r}_{W^{(1)}_\Psi}$, the coefficients $*$ are in $\mathbb{Z}$ and $\widetilde{w} \in (\mathcal{W}_\Psi)^{s+r}$ with $w^{(1)}_{[s_1,t_1]} w^{(1)}_{[s_2+r,t_2]} < \widetilde{w}$. Since $[\zeta^{(1)}_{[s_1,t_1]}, \zeta^{(1)}_{[s_2+r,t_2]}] \in W^{(1),t}_{s+r,\Psi}$, it is (uniquely) written as a $\mathbb{Z}$-linear combination of Lyndon polynomials of the form $\zeta^{(1)}_{[s+r,t]}$. That is,

$$[\zeta^{(1)}_{[s_1,t_1]}, \zeta^{(1)}_{[s_2+r,t_2]}] = \sum_{\rho=1}^{m_{s,r}} \alpha_\rho \zeta^{(1),\rho}_{[s+r,t]} \tag{♭′}$$

with $\alpha_\rho \in \mathbb{Z}$. Without loss the generality, we assume that the corresponding Lyndon words are ordered as $w^{(1),1}_{[s+r,t]} < \cdots < w^{(1),m_{s,t}}_{[s+r,t]}$. Since the Lyndon word $w^{(1)}_{[s_1,t_1]} w^{(1)}_{[s_2+r,t_2]}$, say $w^{(1)}_{[s+r,t]}$, occurs in the right hand side of the equation ($\flat$), it should be occurred in the right hand side of the equation ($\flat'$). By the ordering of $w^{(1),\rho}_{[s+r,t]}$ ($\rho = 1, \ldots, m_{s,t}$), the equation ($\flat$) and Lemma 12, we must have $\alpha_1 = 1$ and $\zeta^{(1),1}_{[s+t,r]} = q^{s+r}_{\mathrm{grad}}(w^{(1)}_{[s+r,t]})$. Having in mind the conditions which are satisfied by $w^{(1)}_{[s_1+r,t_1]}$ and $w^{(1)}_{[s_2+r,t_2]}$, and the equation ($\flat'$), we have

$$[\zeta^{(1)}_{[s,t]}, b] = \zeta^{(1)}_{[s+r,t]} + w'_{s+r,t} \ \text{ with } w^{(1)}_{[s,t]} < w^{(1)}_{[s+r,t]}$$

and $w'_{s+r,t}$ satisfies the conditions of our claim. □

The following technical result can be easily shown by an inductive argument.

Lemma 18 *Let K be a commutative ring with unit and characteristic 0. Let $T(V)$ be the tensor algebra on a free K-module V. Consider $T(V)$ as a Lie algebra in a natural way and $L(V)$ be the Lie subalgebra of $T(V)$ generated by V.*

(I) *For homogeneous elements $a, b \in L(V)$ and $m \in \mathbb{N}$*

$$[a, \ {}_m b] = \sum_{\kappa=0}^{m} (-1)^\kappa \binom{m}{\kappa} b^\kappa a b^{m-\kappa}.$$

(II) *For homogeneous elements $a, b_1, \ldots, b_\tau \in L(V)$ and $m_1, \ldots, m_\tau \in \mathbb{N}$*

$$[a,_{m_1} b_1, \ldots,_{m_\tau} b_\tau] = \sum_{\kappa_\tau=0}^{m_\tau} \cdots \sum_{\kappa_1=0}^{m_1} (-1)^{\kappa_\tau+\cdots+\kappa_1} \binom{m_\tau}{\kappa_\tau} \cdots \binom{m_1}{\kappa_1} b_\tau^{\kappa_\tau} \cdots b_1^{\kappa_1} a b_1^{m_1-\kappa_1} \cdots b_\tau^{m_\tau-\kappa_\tau}.$$

Next, we prove a technical result giving us some information about the Lyndon polynomials over $\mathcal{W}^{(1)}_\Psi$.

Lemma 19 *Let $\mathcal{W}^{(1)}_\Psi = \{z_1, z_2, \ldots\}$ with $z_1 \ll z_2 \ll \cdots$ (as in section 3.3). For $i = 1, \ldots, \tau$, let $w_i \in \mathbb{L}_{W^{(1)}_\Psi} \cap (\mathcal{W}^{(1)}_\Psi)^{r_i}$, $r_i \in \mathbb{N}$, such that $w_\tau < w_{\tau-1} < \cdots < w_1$, and let $w_\zeta = \zeta_1^{\mu_1} \cdots \zeta_\tau^{\mu_\tau}$ where $\zeta_i = q(w_i)$ and $\mu_i \in \mathbb{N}$, $i = 1, \ldots, \tau$. Then, for $z \in \mathcal{W}^{(1)}_\Psi$ with $z \neq \zeta_1$, either $[z, \ {}_{\mu_1}\zeta_1, \ldots, \ {}_{\mu_\tau}\zeta_\tau] = q(zw_1^{\mu_1} \cdots w_\tau^{\mu_\tau})$ with $zw_1^{\mu_1} \cdots w_\tau^{\mu_\tau} \in \mathbb{L}_{W^{(1)}_\Psi}$ or, $[z, \ {}_{\mu_1}\zeta_1, \ldots, \ {}_{\mu_\tau}\zeta_\tau] = w_\tau^{\mu_\tau} \cdots w_1^{\mu_1} z + v$, with $w_\tau^{\mu_\tau} \cdots w_1^{\mu_1} z \in \mathbb{L}_{W^{(1)}_\Psi}$ and where v is a $\mathbb{Z}$-linear combination of words $\widetilde{v} \in \mathcal{W}^+_\Psi$ such that $w_\tau^{\mu_\tau} \cdots w_1^{\mu_1} z < \widetilde{v}$ or, for unique $i \in \{2, \ldots, \tau-1\}$, $[z, \ {}_{\mu_1}\zeta_1, \ldots, \ {}_{\mu_\tau}\zeta_\tau] = w_\tau^{\mu_\tau} \cdots w_i^{\mu_i} z w_1^{\mu_1} \cdots w_{i-1}^{\mu_{i-1}} + v'$, with $w_\tau^{\mu_\tau} \cdots w_i^{\mu_i} z w_1^{\mu_1} \cdots w_{i-1}^{\mu_{i-1}} \in \mathbb{L}_{W^{(1)}_\Psi}$ and where v' is a $\mathbb{Z}$-linear combination of words $\widetilde{v'} \in \mathcal{W}^+_\Psi$ such that $w_\tau^{\mu_\tau} \cdots w_i^{\mu_i} z w_1^{\mu_1} \cdots w_{i-1}^{\mu_{i-1}} < \widetilde{v'}$.*

Proof. Since $(\mathcal{W}^{(1)}_\Psi)^+$ is a totally ordered set (see section 3.3), we have either $z < w_\tau$ or $w_1 < z$ or there exists a unique $i \in \{2, \ldots, \tau-1\}$ such that $w_\tau < \cdots < w_i < z < w_{i-1} < \cdots < w_1$. Suppose that $z < w_\tau$. By Lemma 12, $zw_1^{\mu_1} \in \mathbb{L}_{W^{(1)}_\Psi}$, and its standard factorization is $(zw_1^{\mu_1-1}) \cdot w_1$ (since $z \in \mathcal{W}^{(1)}_\Psi$). Since $z < w_2 < w_1$, we have $(zw_1^{\mu_1})w_2^{\mu_2} \in \mathbb{L}_{W^{(1)}_\Psi}$, and its

standard factorization is $(zw_1^{\mu_1}w_2^{\mu_2-1})\cdot w_2$. Continuing in this way, we obtain $zw_1^{\mu_1}\cdots w_\tau^{\mu_\tau}\in \mathbb{L}_{W_\Psi^{(1)}}$, and its standard factorization is $(zw_1^{\mu_1}\cdots w_\tau^{\mu_\tau-1})\cdot w_\tau$. Thus

$$[z,\ {}_{\mu_{1,4}}\zeta_{1,4},\dots,\ {}_{\mu_{\tau,4}}\zeta_{\tau,4}] = q(zw_{1,4}^{\mu_{1,4}}\cdots w_{\tau(4),4}^{\mu_{\tau(4),4}}).$$

Next, we assume that $w_{1,4}<z$. By Lemma 18,

$$[z,\ {}_{\mu_1}\zeta_1,\dots,\ {}_{\mu_\tau}\zeta_\tau] = (-1)^{\mu_1+\cdots+\mu_\tau}\zeta_\tau^{\mu_\tau}\cdots\zeta_1^{\mu_1}z + z\zeta_1^{\mu_1}\cdots\zeta_\tau^{\mu_\tau} + \sum * \ \zeta_\tau^{\kappa_\tau}\cdots\zeta_1^{\kappa_1}z\zeta_1^{\mu_1-\kappa_1}\cdots\zeta_\tau^{\mu_\tau-\kappa_\tau} \tag{8}$$

with $\kappa_1+\cdots+\kappa_\tau\geq 1$ and $\mu_j-\kappa_j\geq 1$ for at least one $j\in\{1,\dots,\tau\}$, and the coefficients $*$ are in $\mathbb{Z}\setminus\{-1,0,1\}$. By Lemma 12, for $i\in\{1,\dots,\tau\}$, we have

$$\zeta_i = q(w_i) = w_i + \sum_{w_i<\widetilde{w}_i} * \ \widetilde{w}_i \tag{9}$$

where $\widetilde{w}_i\in(\mathcal{W}_\Psi^{(1)})^{r_i}$, $i=1,\dots,\tau$, and the coefficients $*$ are in $\mathbb{Z}$, whose exact values are not important. By the equation (9),

$$\zeta_\tau^{\mu_\tau}\cdots\zeta_1^{\mu_1}z = w_\tau^{\mu_\tau}\cdots w_1^{\mu_1}z + \sum * \ u_1u_2\cdots u_{m_{\lambda,Y}(4)}z \tag{10}$$

where the summation runs over words $u_1,u_2,\dots,u_{m_{\lambda,Y}(4)}$ with the first $u_1,\dots,u_{\mu_\tau}\geq w_\tau$ and $u_1,\dots,u_{\mu_\tau}\in(\mathcal{W}_\Psi^{(1)})^{r_\tau}$, the next $u_{\mu_\tau+1},\dots,u_{\mu_\tau+\mu_{\tau-1}}\geq w_{\tau-1}$ and $u_{\mu_\tau+1},\dots,u_{\mu_\tau+\mu_{\tau-1}}\in(\mathcal{W}_\Psi^{(1)})^{r_{\tau-1}}$ and so on, and $u_j>w_j$ for at least one j; in this case, $u_1u_2\cdots u_{m_{\lambda,Y}(4)}z > w_\tau^{\mu_\tau}\cdots w_1^{\mu_1}z$. Thus, $w_\tau^{\mu_\tau}\cdots w_1^{\mu_1}z$ does not occur in the summation in the right hand side of the equation (10). Similar analysis, we have for $\zeta_\tau^{\kappa_\tau}\cdots\zeta_1^{\kappa_1}z\zeta_1^{\mu_1-\kappa_1}\cdots\zeta_\tau^{\mu_\tau-\kappa_\tau}$ with $\kappa_1+\cdots+\kappa_\tau\geq 1$ and $\mu_j-\kappa_j\geq 1$ for at least one $j\in\{1,\dots,\tau\}$, and for $z\zeta_1^{\mu_1}\cdots\zeta_\tau^{\mu_\tau}$. Since $z>w_1>\cdots>w_\tau$, we have by Lemma 12, $w_\tau^{\mu_\tau}\cdots w_1^{\mu_1}z\in\mathbb{L}_{W_\Psi^{(1)}}$. By the above discussion, the Lyndon word $w_\tau^{\mu_\tau}\cdots w_1^{\mu_1}z$ is the smallest (in the ordering of $(\mathcal{W}_\Psi^{(1)})^+$) of the words occurring in the equation (8). Finally, we assume that there exists a unique $i\in\{2,\dots,\tau-1\}$ such that

$$w_\tau<\cdots<w_i<z<w_{i-1}<\cdots<w_1. \tag{11}$$

Since $z<w_{i-1}<\cdots<w_1$, we obtain, as in the case $z<w_\tau$, $zw_1^{\mu_1}\cdots w_{i-1}^{\mu_{i-1}}\in\mathbb{L}_{W_\Psi^{(1)}}$, and its standard factorization is $(zw_1^{\mu_1}\cdots w_{i-1}^{\mu_{i-1}-1})\cdot w_{i-1}$. Thus

$$[z,\ {}_{\mu_1}\zeta_1,\dots,\ {}_{\mu_{i-1}}\zeta_{i-1}] = q(zw_1^{\mu_1}\cdots w_{i-1}^{\mu_{i-1}}).$$

For the next few lines, we write $a=q(zw_1^{\mu_1}\cdots w_{i-1}^{\mu_{i-1}})$ and $w=zw_1^{\mu_1}\cdots w_{i-1}^{\mu_{i-1}}$. By the equation (11), $w_\tau^{\mu_\tau}\cdots w_i^{\mu_i}w\in\mathbb{L}_{W_\Psi^{(1)}}$. By Lemma 18,

$$[a,\ {}_{\mu_i}\zeta_i,\dots,\ {}_{\mu_\tau}\zeta_\tau] = (-1)^{\mu_i+\cdots+\mu_\tau}\zeta_\tau^{\mu_\tau}\cdots\zeta_1^{\mu_1}a + a\zeta_i^{\mu_i}\cdots\zeta_\tau^{\mu_\tau} + \sum * \ \zeta_\tau^{\kappa_\tau}\cdots\zeta_i^{\kappa_i}a\zeta_i^{\mu_i-\kappa_i}\cdots\zeta_\tau^{\mu_\tau-\kappa_\tau} \tag{12}$$

with $\kappa_i+\cdots+\kappa_\tau\geq 1$ and $\mu_j-\kappa_j\geq 1$ for at least one $j\in\{i,\dots,\tau\}$, and the coefficients $*$ are in $\mathbb{Z}\setminus\{-1,0,1\}$. By applying similar arguments as in the case $w_1<z$, we obtain the Lyndon word $w_\tau^{\mu_\tau}\cdots w_i^{\mu_i}w$ is the smallest of the words occurring in the equation (12). $\square$

5.8 The main result

In this subsection we show one of the main results of this paper. It gives us a $\mathbb{Z}$-module decomposition of J^{c+2} in terms of J_C^{c+2} helping us to deduce L^{c+2}/J^{c+2} is torsion-free.

Before stating the main result and giving its proof, we need to recall some notation and definitions. Let $c \geq 2$, and $\lambda = (c^{n(c)}, (c-1)^{n(c-1)}, \ldots, 2^{n(2)}, 1^{n(1)}) \in \mathrm{Part}(c)$. Each element $w(p_{m_{\lambda,Y}(t)})$ (with $t = 1, 2, 3$) has the form

$$w(p_{m_{\lambda,Y}(t)}) = a_{1,t}^{(1)} \cdots a_{1,t}^{(\nu_{t,1})} a_{2,t}^{(1)} \cdots a_{2,t}^{(\nu_{t,2})} \cdots a_{d,t}^{(1)} \cdots a_{d,t}^{(\nu_{t,d})},$$

where $a_{1,t}^{(1)} \preceq \cdots \preceq a_{1,t}^{(\nu_{t,1})} \preceq \cdots \preceq a_{d,t}^{(1)} \preceq \cdots \preceq a_{d,t}^{(\nu_{t,d})}$ with $a_{r,t}^{(1)}, \ldots, a_{r,t}^{(\nu_{t,r})} \in q^r(\mathbb{L}_{V_t})$ $(r = 1, \ldots, d)$ with $d \leq c$. Let us assume that $\nu_{t,r} \geq 1$ (for $r = 1, \ldots, d$). Then, there are unique Lyndon words $w_{r,t}^{(1)}, \ldots, w_{r,t}^{(\nu_{t,r})} \in \mathbb{L}_{V_t}^r$ $(r = 1, \ldots, d)$ such that $q(w_{r,t}^{(s)}) = a_{r,t}^{(s)}$, $r = 1, \ldots, t$, $s = 1, \ldots, \nu_{t,r}$. By Lemma 12 (for $A = \mathcal{V}_t$) each $q(w_{r,t}^{(s)}) = w_{r,t}^{(s)} + v_{r,t}^{(s)}$, where $v_{r,t}^{(s)}$ belongs to the $\mathbb{Z}$-submodule of $T(V_t)$ spanned by those words $\widetilde{v}_{r,t}^{(s)} \in \mathcal{V}_t^r$ such that $w_{r,t}^{(s)} < \widetilde{v}_{r,t}^{(s)}$. Note that for $r = 1$, $w_{1,t}^{(s)} = a_{1,t}^{(s)} \in \mathcal{V}_t$, $s = 1, \ldots, \nu_{t,1}$ (and so, $v_{1,t}^{(s)}$ is the empty word, $s = 1, \ldots, \nu_{t,1}$). Define the corresponding word of $w(p_{m_{\lambda,Y}(t)})$ as

$$w_\ell(p_{m_{\lambda,Y}(t)}) = w_{1,t}^{(1)} \cdots w_{1,t}^{(\nu_{t,1})} w_{2,t}^{(1)} \cdots w_{2,t}^{(\nu_{t,2})} \cdots w_{d,t}^{(1)} \cdots w_{d,t}^{(\nu_{t,d})} \in \mathcal{V}_t^{m_{\lambda,Y}(t)}.$$

Thus we may write

$$w(p_{m_{\lambda,Y}(t)}) = b_{1,t}^{\mu_{1,t}} b_{2,t}^{\mu_{2,t}} \cdots b_{\tau(t),t}^{\mu_{\tau(t),t}} \tag{13}$$

with $\mu_{1,t}, \ldots, \mu_{\tau(t),t}$ positive integers and $b_{i,t} \in q^{r_{i,t}}(\mathbb{L}_{V_t})$, $i = 1, \ldots, \tau(t)$ and $r_{i,t} \leq r_{j,t}$ with $i < j$ and so, the corresponding word $w_\ell(p_{m_{\lambda,Y}(t)})$ is written $w_\ell(p_{m_{\lambda,Y}(t)}) = w_{1,t}^{\mu_{1,t}} \cdots w_{\tau(t),t}^{\mu_{\tau(t),t}}$ where $b_{i,t} = q(w_{i,t})$, $i = 1, \ldots, \tau(t)$. Having in mind the basic property of Lyndon words (see Lemma 12 for $A = \mathcal{V}_t$) we rearrange the elements in the expression (13) as follows: there exists a unique $\pi \in \mathrm{Sym}(\tau(t))$ such that $w_{\pi(\tau(t)),t} < \cdots < w_{\pi(1),t}$ in the lexicographical order and so, the desired rearrangement of $w(p_{m_{\lambda,Y}(t)})$ is $w_\pi(p_{m_{\lambda,Y}(t)}) = b_{\pi(1),t}^{\mu_{\pi(1),t}} b_{\pi(2),t}^{\mu_{\pi(2),t}} b_{\pi(3),t}^{\mu_{\pi(3),t}} \cdots b_{\pi(\tau(t)),t}^{\mu_{\pi(\tau(t)),t}}$. For simplicity, we write

$$w(p_{m_{\lambda,Y}(t)}) = b_{1,t}^{\mu_{1,t}} b_{2,t}^{\mu_{2,t}} b_{3,t}^{\mu_{3,t}} \cdots b_{\tau(t),t}^{\mu_{\tau(t),t}}, \quad w_\ell(p_{m_{\lambda,Y}(t)}) = w_{1,t}^{\mu_{1,t}} w_{2,t}^{\mu_{2,t}} w_{3,t}^{\mu_{3,t}} \cdots w_{\tau(t),t}^{\mu_{\tau(t),t}} \tag{14}$$

with $w_{\tau(t),t} < w_{\tau(t)-1,t} < \cdots < w_{1,t}$. Replacing each $b_{r,t}$, occurring in the equation (14), by $w_{r,t} + v_{r,t}$, we obtain from Corollary 2

$$w(p_{m_{\lambda,Y}(t)}) = w_\ell(p_{m_{\lambda,Y}(t)}) + \sum_{\text{finite}} * \, w_{f,\ell}(p_{m_{\lambda,Y}(t)}), \tag{15}$$

where the coefficients $*$ are in $\mathbb{Z}$, $w_\ell(p_{m_{\lambda,Y}(t)}) < w_{f,\ell}(p_{m_{\lambda,Y}(t)})$ in the alphabetical order for all $w_{f,\ell}(p_{m_{\lambda,Y}(t)})$ appearing in the sum of the equation (15), and

$$w_{f,\ell}(p_{m_{\lambda,Y}(t)}) = f_{1,t}^{\mu_{1,t}} f_{2,t}^{\mu_{2,t}} f_{3,t}^{\mu_{3,t}} \cdots f_{\tau(t),t}^{\mu_{\tau(t),t}}$$

with $f_{s,t} \in \{w_{s,t}, \widetilde{v}_{s,t}\}$, $s = 1, \ldots, \tau(t)$, and at least one $f_{s,t}$ is equal to $\widetilde{v}_{s,t}$ in the aforementioned expression. For $\kappa = 1, 2, 3$, $[L^2(V_\kappa); \Phi_\lambda/\Phi_{\lambda+1}]$ is the $\mathbb{Z}$-submodule of $[J^2; T^c]$ spanned

by all Lie commutators of the form $[y_\kappa; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})]$ with $Y \in \mathcal{G}_\lambda$. We introduce some notation: Fix $Y \in \mathcal{G}_\lambda$. For $t_1 \in \{1,2,3\}$, with $m_{\lambda,Y}(t_1) \geq 1$, we write

$$w_{\lambda,Y}(y_\kappa; t_1) = [y_\kappa; w(p_{m_{\lambda,Y}(t_1)})] = [y_\kappa, \ {}_{\mu_{1,t_1}}b_{1,t_1}, \ {}_{\mu_{2,t_1}}b_{2,t_1}, \ldots, \ {}_{\mu_{\tau(t_1),t_1}}b_{\tau(t_1),t_1}].$$

For $t_2 \in \{1,2,3\} \setminus \{t_1\}$ with $m_{\lambda,Y}(t_1), m_{\lambda,Y}(t_2) \geq 1$, we write

$$w_{\lambda,Y}(y_\kappa; t_1, t_2) = [w_{\lambda,Y}(y_\kappa; t_1); w(p_{m_{\lambda,Y}(t_2)})] = [w_{\lambda,Y}(y_\kappa; t_1), \ {}_{\mu_{1,t_2}}b_{1,t_2}, \ldots, \ {}_{\mu_{\tau(t_2),t_2}}b_{\tau(t_2),t_2}]$$

and, for $t_3 \neq t_1, t_2$ with $m_{\lambda,Y}(t_3) \geq 1$, $w_{\lambda,Y}(y_\kappa; t_1, t_2, t_3) = [w_{\lambda,Y}(y_\kappa; t_1, t_2); w(p_{m_{\lambda,Y}(t_3)}]$. Similarly, we write for $w_{\lambda,Y,\ell}(y_\kappa; t_1)$, $w_{\lambda,Y,\ell}(y_\kappa; t_1, t_2)$, $w_{\lambda,Y,\ell}(y_\kappa; t_1, t_2, t_3)$, $w_{\lambda,Y,\ell,f}(y_\kappa; t_1)$, $w_{\lambda,Y,\ell,f}(y_\kappa; t_1, t_2)$ and $w_{\lambda,Y,\ell,f}(y_\kappa; t_1, t_2, t_3)$. Furthermore, for $u \in L'$ and $v = z_1 \cdots z_m \in \bigcup_{j=1}^{3} \mathcal{V}_j^+$, we write $f(u; v) = [u, z_1, \ldots, z_m]$. It is clearly enough that $f(u; v_1 v_2) = f(f(u; v_1); v_2)$ for all $u \in L'$ and $v_1, v_2 \in \bigcup_{j=1}^{3} \mathcal{V}_j^+$.

Our main result in this section is the following.

Theorem 2 *Let $c \geq 2$. There are $\mathbb{Z}$-submodules U_1^{c+2}, U_2^{c+2}, U_3^{c+2} and U_Ψ^{c+2} of J_1^{c+2}, J_2^{c+2}, J_3^{c+2} and J_Ψ^{c+2}, respectively, such that*

$$J^{c+2} = U_1^{c+2} \oplus U_2^{c+2} \oplus U_3^{c+2} \oplus U_\Psi^{c+2} \oplus J_C^{c+2} = (J \cap L_{\text{grad}}^{c+2}(W_\Psi^{(1)})) \oplus J_C^{c+2}.$$

Furthermore, $J \cap L_{\text{grad}}^{c+2}(W_\Psi^{(1)})$ is a direct summand of $L_{\text{grad}}^{c+2}(W_\Psi^{(1)})$. In particular,

$$L^{c+2} = (L_{\text{grad}}^{c+2}(W_\Psi^{(1)}))^* \oplus J^{c+2},$$

where $L_{\text{grad}}^{c+2}(W_\Psi^{(1)}) = (L_{\text{grad}}^{c+2}(W_\Psi^{(1)}))^* \oplus (J \cap L_{\text{grad}}^{c+2}(W_\Psi^{(1)}))$.

Proof. We shall prove our claim into four cases.

Case 1. An analysis of $J_{1,\lambda}^{c+2}$ modulo $(J_{1,\lambda+1}^{c+2} + J_C^{c+2})$. Recall that, for $\lambda \in \text{Part}(c)$,

$$J_{1,\lambda}^{c+2} + J_C^{c+2} = [L^2(V_1); \Phi_\lambda/\Phi_{\lambda+1}] + J_{1,\lambda+1}^{c+2} + J_C^{c+2}$$

and

$$[L^2(V_1); \Phi_\lambda/\Phi_{\lambda+1}] = \mathbb{Z} - \text{span}\{[y_1; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})], Y \in \mathcal{G}_\lambda\}.$$

In the next subcases, we analyze the Lie commutator $[y_1; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})]$ with $Y \in \mathcal{G}_\lambda$. So, we fix $Y \in \mathcal{G}_\lambda$.

Subcase 1a. $m_{\lambda,Y}(1) \geq 1$. Since $w_{\lambda,Y}(y_1; 1) \neq 0$, we have $x_1 x_2 \neq w_{1,1}$. By the equation (15) (for $t = 1$), we get

$$w_{\lambda,Y}(y_1; 1) = w_{\lambda,Y,\ell}(y_1; 1) + \sum_{\text{finite}} * \ w_{\lambda,Y,\ell,f}(y_1; 1).$$

If each $m_{\lambda,Y}(j) = 0$ for $j = 2, \ldots, 6$, we have $[y_1; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})] = w_{\lambda,Y}(y_1; 1) \in J_C^{c+2}$. Thus we assume $m_{\lambda,Y}(j) \geq 1$ for some $j \in \{2, \ldots, 6\}$. Suppose that $m_{\lambda,Y}(2) \geq 1$. If $w_{1,2} = x_4$, then $w_\ell(p_{m_{\lambda,Y}(2)}) = x_4^{\mu_{1,2}}$ and $w(p_{m_{\lambda,Y}(2)}) = x_4^{\mu_{1,2}}$. Using the Jacobi identity in the form $[x, y, z] = [x, z, y] + [x, [y, z]]$ and since $\psi_2(y_7) = y_7 + y_5$, we have

$$[w_{\lambda,Y}(y_1; 1), x_4] = f([y_1, x_4]; w_\ell(p_{m_{\lambda,Y}(1)})) + w,$$

where $w \in L_{\text{grad,e}}^{3+m_{\lambda,Y}(1)}(W_\Psi)$ with $e \geq 2$. By the appropriate equation of (C1),

$$[w_{\lambda,Y}(y_1;1),\ {}_{\mu_{1,2}}x_4] = f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1}) + f([y_5,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1})$$
$$-f([\psi_2(y_7),x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1}) + [w,\ {}_{(\mu_{1,2}-1)}x_4]. \qquad (16)$$

We notice that $[w,\ {}_{(\mu_{1,2}-1)}x_4] = w^{(1)}_{(1,4)} + w^{(2)}_{(1,4)} + \widetilde{w}_{(1,4)}$ where $w^{(\kappa)}_{(1,4)} \in L^{2+m_{\lambda,Y}(1)+\mu_{1,2}}_{\mathrm{grad},e_\kappa}(W^{(\kappa)}_\Psi)$ ($e_\kappa \geq 2, \kappa = 1,2$) and $\widetilde{w}_{(1,4)} \in L^{2+m_{\lambda,Y}(1)+\mu_{1,2}}_{\mathrm{grad},e_\kappa}(\widetilde{W}_{\Psi,J})$. Furthermore,

$$f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1}), f([y_5,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1}) \in \mathcal{W}^{(1)}_{2+m_{\lambda,Y}(1)+\mu_{1,2},\Psi},$$
$$f([\psi_2(y_7),x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1}) \in \mathcal{W}^{(2)}_{2+m_{\lambda,Y}(1)+\mu_{1,2},\Psi},$$

and

$$f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1})+f([y_5,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1})+w^{(1)}_{(1,4)} \in J\cap L^{2+m_{\lambda,Y}(1)+\mu_{1,2}}_{\mathrm{grad}}(W^{(1)}_\Psi).$$

Thus, we assume that $w_{1,2} \neq x_4$. Each element in $\mathbb{L}_{V_2}$, but not x_4, starts with x_3 and ends with x_4. Using the Jacobi identity as before, the suitable equations of (C1) and $y_6(= [x_3,x_2]) = \psi_2(y_6) - \psi_2(y_7) + y_5$,

$$\begin{aligned}
w_{\lambda,Y}(y_1;1,2) \quad = \quad & -f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}})+\\
& f([y_4,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) -\\
& f([\psi_2(y_6),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) +\\
& f([\psi_2(y_7),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) + w^{(1,2)}, \qquad (17)
\end{aligned}$$

where $w'_{1,2}$ is the unique element in $\mathcal{V}_2^*$ (the free monoid on $\mathcal{V}_2$) such that $w_{1,2} = x_3w'_{1,2}$, and $w^{(1,2)} \in L^{\rho_{\lambda,Y}(1,2)+2}_{\mathrm{grad,e}}(W_\Psi)$ with $e \geq 2$ and $\rho_{\lambda,Y}(1,2) = m_{\lambda,Y}(1) + m_{\lambda,Y}(2)$. In particular, $w^{(1,2)} = w^{(1,2,1)}_{(1,3)} + w^{(1,2,2)}_{(1,3)} + \widetilde{w}^{(1,2)}_{(1,3)}$, where $w^{(1,2,\kappa)}_{(1,3)} \in L^{2+\rho_{\lambda,Y}(1,2)}_{\mathrm{grad},e_\kappa}(W^{(\kappa)}_\Psi)$ ($e_\kappa \geq 2, \kappa = 1,2$) and $\widetilde{w}^{(1,2)}_{(1,3)} \in L^{2+\rho_{\lambda,Y}(1,2)}_{\mathrm{grad,e}}(\widetilde{W}_{\Psi,J})$, $e \geq 2$. Note that

$$\begin{aligned}
& f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}),\\
& f([y_4,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) \in \mathcal{W}^{(1)}_{2+\rho_{\lambda,Y}(1,2),\Psi},
\end{aligned}$$

$$\begin{aligned}
& f([\psi_2(y_6),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}),\\
& f([\psi_2(y_7),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) \in \mathcal{W}^{(2)}_{2+\rho_{\lambda,Y}(1,2),\Psi},
\end{aligned}$$

and

$$\begin{aligned}
& -f([y_5,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) \quad + \\
& f([y_4,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)}\ldots w_{\tau(2),2}^{\mu_{\tau(2),2}}) \quad + \\
& \qquad w^{(1,2,1)}_{(1,3)} \in J \cap L^{2+\rho_{\lambda,Y}(1,2)}_{\mathrm{grad}}(W^{(1)}_\Psi).
\end{aligned}$$

If $m_{\lambda,Y}(3) \geq 1$ and $m_{\lambda,Y}(2) = 0$, then, by applying similar arguments as above and $y_9 = \psi_2(y_9) - y_8$, we have by the suitable equations of (C1),

$$[w_{\lambda,Y}(y_1;1),\ {}_{\mu_{1,3}}x_6] = -f([y_{11},x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}) - f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1})+$$

$$-f([\psi_2(y_9),x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}) + [w_1,\ {}_{(\mu_{1,3}-1)}x_6], \qquad (18)$$

where $w_1 \in L_{\text{grad,e}}^{3+m_{\lambda,Y}(1)}(W_\Psi)$ with $e \geq 2$. In particular, $[w_1,\ {}_{(\mu_{1,3}-1)}x_6] = w_{(1,6)}^{(1)} + w_{(1,6)}^{(2)} + \widetilde{w}_{(1,6)}$ where $w_{(1,6)}^{(\kappa)} \in L_{\text{grad,e}_\kappa}^{2+m_{\lambda,Y}(1)+\mu_{1,3}}(W_\Psi^{(\kappa)})$ ($e_\kappa \geq 2, \kappa = 1,2$) and $\widetilde{w}_{(1,6)} \in L_{\text{grad,e}}^{2+m_{\lambda,Y}(1)+\mu_{1,3}}(\widetilde{W}_{\Psi,J})$, $e \geq 2$. Note that

$$f([y_{11},x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}),\ f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}) \in \mathcal{W}_{m_{\lambda,Y}(1)+\mu_{1,3}+2,\Psi}^{(1)}$$

and

$$f([\psi_2(y_9),x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}) \in \mathcal{W}_{m_{\lambda,Y}(1)+\mu_{1,3}+2,\Psi}^{(2)}.$$

Furthermore,

$$-f([y_{11},x_1];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1}) - f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})x_6^{\mu_{1,3}-1})+$$

$$w_{(1,6)}^{(1)} \in J \cap L_{\text{grad}}^{2+m_{\lambda,Y}(1)+\mu_{1,3}}(W_\Psi^{(1)}).$$

Thus, we assume that $w_{1,3} \neq x_6$. Each element in $\mathbb{L}_{V_5}$, but not x_6, starts with x_5 and ends with x_6. By using the Jacobi identity as before, $y_{10} = \psi_2(y_{10}) - y_8$ and the suitable equations of (C1), we have

$$\begin{aligned} w_{\lambda,Y}(y_1;1,3) \quad = \quad & f([y_8,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) - \\ & f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) + \\ & -f([\psi_2(y_{10}),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) + w^{(1,3)}, \qquad (19) \end{aligned}$$

where $w'_{1,3}$ is the unique element in $\mathcal{V}_3^*$ (the free monoid on $\mathcal{V}_3$) such that $w_{1,3} = x_5 w'_{1,3}$, and $w^{(1,3)} \in L_{\text{grad}}^{\rho_{\lambda,Y}(1,3)+2}(W_\Psi)$ with $\rho_{\lambda,Y}(1,3) = m_{\lambda,Y}(1) + m_{\lambda,Y}(3)$. In fact, $w^{(1,3)} = w_1^{(1,3)} + w_2^{(1,3)} + w_{\Psi,J}^{(1,3)}$ where $w_\kappa^{(1,3)} \in L_{\text{grad,e}_\kappa}^{\rho_{\lambda,Y}(1,3)+2}(W_\Psi^{(\kappa)})$ ($e_\kappa \geq 2, \kappa = 1,2$) and $w^{(1,3)} \in L_{\text{grad,e}}^{\rho_{\lambda,Y}(1,3)+2}(\widetilde{W}_{\Psi,J})$, $e \geq 2$. Note that

$$\begin{aligned} & f([y_8,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}), \\ & f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) \in \mathcal{W}_{\rho_{\lambda,Y}(1,3)+2,\Psi}^{(1)}, \end{aligned}$$

$$f([\psi_2(y_{10}),x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}w_{1,3}^{(\mu_{1,3}-1)}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) \in \mathcal{W}_{\rho_{\lambda,Y}(1,3)+2,\Psi}^{(2)}$$

and

$$f([y_8,x_1];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}}) - f([y_8,x_2];w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,3}\dots w_{\tau(3),3}^{\mu_{\tau(3),3}})+$$

$$w_1^{(1,3)} \in J \cap L_{\text{grad}}^{\rho_{\lambda,Y}(1,3)+2}(W_\Psi^{(1)}).$$

¿From the equations (16) and (15) (for $t = 3$),

$$f([w_{\lambda,Y}(y_1;1),\ {}_{\mu_{1,2}}x_4]; w_\ell(p_{\lambda,Y}(3))) = f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) +$$
$$f([y_5, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) -$$
$$f([\psi_2(y_7), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) + w_2, \tag{20}$$

where $w_2 \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad,e}}(W_\Psi)$, $e \geq 2$, $\rho_{\lambda,Y}(1,2,3) = m_{\lambda,Y}(1) + m_{\lambda,Y}(2) + m_{\lambda,Y}(3)$. In particular, $w_2 = w_2^{(1)} + w_2^{(2)} + \widetilde{w}_2$, where $w_2^{(\kappa)} \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad},e_\kappa}(W_\Psi^{(\kappa)})$ $(e_\kappa \geq 2, \kappa = 1,2)$ and $\widetilde{w}_2 \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad},e}(\widetilde{W}_{\Psi,J})$. The elements

$$f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))),$$
$$f([y_5, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) \in \mathcal{W}^{(1)}_{\rho_{\lambda,Y}(1,2,3)+2,\Psi},$$

$$f([\psi_2(y_7), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) \in \mathcal{W}^{(2)}_{\rho_{\lambda,Y}(1,2,3)+2,\Psi}$$

and

$$f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3)))+$$
$$f([y_5, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})x_4^{\mu_{1,2}-1} w_\ell(p_{\lambda,Y}(3))) + w_2^{(1)} \in J \cap L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad}}(W_\Psi^{(1)}).$$

Finally, from the equations (17) and (15) (for $t = 3$), we have

$$w_{\lambda,Y}(y_1;1,2,3) = -f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}})+$$
$$f([y_4, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}})-$$
$$f([\psi_2(y_6), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}})+$$
$$f([\psi_2(y_7), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}}) + w^{(1,2,3)}, \tag{21}$$

where $w^{(1,2,3)} \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad,e}}(W_\Psi)$, $e \geq 2$. In particular, $w^{(1,2,3)} = w_1^{(1,2,3)} + w_2^{(1,2,3)} + \widetilde{w}^{(1,2,3)}$, where $w_\kappa^{(1,2,3)} \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad},e_\kappa}(W_\Psi^{(\kappa)})$ $(e_\kappa \geq 2, \kappa = 1,2)$ and $\widetilde{w}^{(1,2,3)} \in L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad},e}(\widetilde{W}_{\Psi,J})$. The elements

$$f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}}),$$

$$f([y_4, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}}) \in \mathcal{W}^{(1)}_{2+\rho_{\lambda,Y}(1,2,3),\Psi},$$

$$f([\psi_2(y_6), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}}),$$

$$f([\psi_2(y_7), x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}}) \in \mathcal{W}^{(2)}_{2+\rho_{\lambda,Y}(1,2,3),\Psi},$$

and

$$-f([y_5, x_1]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}})+$$
$$f([y_4, x_2]; w_\ell(p_{m_{\lambda,Y}(1)})w'_{1,2}w_{1,2}^{(\mu_{1,2}-1)} \ldots w_{\tau(2),2}^{\mu_{\tau(2),2}} w_{1,3}^{\mu_{1,3}} \ldots w_{\tau(3),3}^{\mu_{\tau(3),3}})+$$
$$w_1^{(1,2,3)} \in J \cap L^{2+\rho_{\lambda,Y}(1,2,3)}_{\text{grad}}(W_\Psi^{(1)}).$$

If $m_{\lambda,Y}(5)$ or $m_{\lambda,Y}(6) \geq 1$, we obtain from the equations $(16)-(21)$ and, by the construction of $L(\widetilde{W}_{\Psi,J})$, the corresponding Lie commutator $[y_1; w(p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}$. Thus we assume that $m_{\lambda,Y}(5) = m_{\lambda,Y}(6) = 0$. Let $m_{\lambda,Y}(4) \geq 1$. Without loss of generality, we consider the case where both $m_{\lambda,Y}(2)$ and $m_{\lambda,Y}(3) \geq 1$. (We deal with the same method the cases either $m_{\lambda,Y}(2) \geq 1$ or $m_{\lambda,Y}(3) \geq 1$.) Thus, by the equations (20), (21), we may write

$$w_{\lambda,Y}(y_1; 1, 2, 3) = \pm v_1 + v_2 + w_{(1,2)},$$

with $v_1, v_2 \in \mathcal{W}_{m,\Psi}^{(1)}$ (that is, v_1, v_2 are Lyndon words of length 1) with $m = 2+\rho_{\lambda,Y}(1,2,3)$ and $w_{(1,2)} \in W_{\Psi}^{(2)} \oplus L_{\text{grad,e}}^{m}(W_{\Psi})$, $e \geq 2$. In particular, $w_{(1,2)} = w_{(1,2)}^{(1)} + w_{(1,2)}^{(1,2)} + w_{(1,2)}^{(2)} + \widetilde{w}_{(1,2)}$ where $w_{(1,2)}^{(\kappa)} \in L_{\text{grad},e_\kappa}^{m}(W_{\Psi}^{(\kappa)})$ ($e_\kappa \geq 2, \kappa = 1, 2$), $w_{(1,2)}^{(1,2)} \in \mathcal{W}_{m,\Psi}^{(2)}$ and $\widetilde{w}_{(1,2)} \in L_{\text{grad},e}^{m}(\widetilde{W}_{\Psi,J})$, $e \geq 2$. Recall that $w(p_{m_{\lambda,Y}(4)}) = \zeta_{1,4}^{\mu_{1,4}} \zeta_{2,4}^{\mu_{2,4}} \cdots \zeta_{\tau(4),4}^{\mu_{\tau(4),4}}$ with $q(w_{i,4}) = \zeta_{i,4}$ and $w_{\tau(4),4} < \cdots < w_{1,4}$ (where $<$ denotes the alphabetical order in $(\mathcal{W}_{\Psi}^{(1)})^{+}$), and $w_{i,4} \in (\mathcal{W}_{\Psi}^{(1)})^{r_i}$ with $r_i \in \mathbb{N}$ and $i = 1, \ldots, \tau(4)$. So,

$$\begin{aligned}
[w_{\lambda,Y}(y_1; 1, 2, 3); w(p_{m_{\lambda,Y}(4)})] \;=\;& \pm[v_1; w(p_{m_{\lambda,Y}(4)})] + [v_2; w(p_{m_{\lambda,Y}(4)})] + \\
& [w_{(1,2)}^{(1)}; w(p_{m_{\lambda,Y}(4)})] + [w_{(1,2)}^{(2)}; w(p_{m_{\lambda,Y}(4)})] + \\
& [w_{(1,2)}^{(1,2)}; w(p_{m_{\lambda,Y}(4)})] + [\widetilde{w}_{(1,2)}; w(p_{m_{\lambda,Y}(4)})].
\end{aligned}$$

Note that $[w_{(1,2)}^{(1)}; w(p_{m_{\lambda,Y}(4)})] \in L_{\text{grad,e}}^{m+m_{\lambda,Y}(4)}(W_{\Psi}^{(1)})$, $e \geq 2+\rho$, where ρ is the minimum of the degrees of $w_{1,4}, \ldots, w_{\tau(4),4}$ in terms of elements of $\mathcal{W}_{\Psi}^{(1)}$, and

$$\pm[v_1; w(p_{m_{\lambda,Y}(4)})] + [v_2; w(p_{m_{\lambda,Y}(4)})] + [w_{(1,2)}^{(1)}; w(p_{m_{\lambda,Y}(4)})] \in J \cap L_{\text{grad}}^{m+m_{\lambda,Y}(4)}(W_{\Psi}^{(1)}).$$

Now either $v_1 = \zeta_{1,4}$ or $v_2 = \zeta_{1,4}$ or $v_1, v_2 \neq \zeta_{1,4}$. Let us assume that $v_1, v_2 \neq \zeta_{1,4}$. Then, by Lemma 19, different Lyndon words occur in the expressions of $[v_1; w(p_{m_{\lambda,Y}(4)})]$ and $[v_2; w(p_{m_{\lambda,Y}(4)})]$. We replace the (unique) Lyndon polynomial corresponding to the smallest Lyndon word (by means of the ordering of $\mathcal{W}_{\Psi}^{(1)}$) by $[w_{\lambda,Y}(y_1; 1, 2, 3); w(p_{m_{\lambda,Y}(4)})]$. Similar arguments may be applied if $v_1 = \zeta_{1,4}$ or $v_2 = \zeta_{1,4}$. Furthermore, we deal with similar arguments the cases either $m_{\lambda,Y}(2) \geq 1$ or $m_{\lambda,Y}(3) \geq 1$. We write $W_{\lambda,Y}(y_1; 1, m_{\lambda,Y}(2) + m_{\lambda,Y}(3), 4)$ for the $\mathbb{Z}$-submodule of $J_{1,\lambda}^{c+2}$ spanned by all Lie commutators of the form $[w_{\lambda,Y}(y_1; 1, 2, 3); w(p_{m_{\lambda,Y}(4)})]$ mentioned above with $m_{\lambda,Y}(1) \geq 1$, $m_{\lambda,Y}(2) + m_{\lambda,Y}(3) \geq 1$ and $m_{\lambda,Y}(4)$.

Next we assume that $m_{\lambda,Y}(2) = m_{\lambda,Y}(3) = 0$ and $m_{\lambda,Y}(4) \geq 1$. Recall that

$$[w_{\lambda,Y}(y_1; 1), w(p_{m_{\lambda,Y}(4)})] = [w_{\lambda,Y}(y_1; 1), {}_{\mu_{1,4}}\zeta_{1,4}, \ldots, {}_{\mu_{\tau(4),4}}\zeta_{\tau(4),4}].$$

Note that $w_{\lambda,Y}(y_1; 1) \in L^{2+m_{\lambda,Y}(1)}(V_1)$. Thus

$$[w_{\lambda,Y}(y_1; 1), w(p_{m_{\lambda,Y}(4)})] = -[\zeta_{1,4}, w_{\lambda,Y}(y_1; 1), {}_{(\mu_{1,4}-1)}\zeta_{1,4}, \ldots, {}_{\mu_{\tau(4),4}}\zeta_{\tau(4),4}].$$

If $m_{\lambda,Y}(5)$ or $m_{\lambda,Y}(6) \geq 1$, then, by Lemma 17 (for $\kappa = \mu = 1$), we have

$$[w_{\lambda,Y}(y_1; 1), w(p_{m_{\lambda,Y}(4)}), w(p_{m_{\lambda,Y}(5)}), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

Thus we assume that $m_{\lambda,Y}(5) = m_{\lambda,Y}(6) = 0$. Let $W_{\lambda,Y}(y_1; 1, 4)$ be the $\mathbb{Z}$-submodule of $J_{1,\lambda}^{c+2}$ spanned by all Lie commutators of the form $[w_{\lambda,Y}(y_1; 1), w(p_{m_{\lambda,Y}(4)})]$. Note that the

Lie commutators of the form $[w_{\lambda,Y}(y_1;1), w(p_{m_{\lambda,Y}(4)})]$ are not effected by the equations $(C1)$, and

$$[w_{\lambda,Y}(y_1;1), w(p_{m_{\lambda,Y}(4)})] \in J \cap L_{\text{grad}}^{2+m_{\lambda,Y}(1)+m_{\lambda,Y}(4)}(W_\Psi^{(1)}).$$

By the analysis of $[y_1, w(p_{m_{\lambda,Y}(1)}), w(p_{m_{\lambda,Y}(2)}), w(p_{m_{\lambda,Y}(3)})]$ in the equations (16)-(21) and the definition of $W_{\lambda,Y}(y_1;1,4)$ (having in mind Lemma 17 (for $\kappa=\mu=1$) and the ordering on $\mathcal{W}_\Psi$ (section 3.3)), we have the sum of $W_{\lambda,Y}(y_1;1,m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)$ and $W_{\lambda,Y}(y_1;1,4)$ is direct.

Let $m_{\lambda,Y}(4)=0$ and $m_{\lambda,Y}(5)\geq 1$. If $m_{\lambda,Y}(2)$ or $m_{\lambda,Y}(3)\geq 1$, then, by the equations (16)-(21), we have

$$[w_{\lambda,Y}(y_1;t_1,t_2,t_3), w(p_{m_{\lambda,Y}(5)}), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

So, we assume that $m_{\lambda,Y}(2)=m_{\lambda,Y}(3)=0$. By Lemma 17 (for $\kappa=2,\mu=1$), we get

$$[w_{\lambda,Y}(y_1;1), w(p_{m_{\lambda,Y}(5)}), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

Finally, we assume that $(m_{\lambda,Y}(4)=)\ \ m_{\lambda,Y}(5)=0$. Suppose that $m_{\lambda,Y}(6)\geq 1$. If $m_{\lambda,Y}(2)$ or $m_{\lambda,Y}(3)\geq 1$, then, by the equations (16)-(21), and since $L(\widetilde{W}_{\Psi,J})$ is an ideal $L(W_\Psi)$, we have

$$[w_{\lambda,Y}(y_1;t_1,t_2,t_3), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

Thus, $m_{\lambda,Y}(2)=m_{\lambda,Y}(3)=0$. By the construction of $L(\widetilde{W}_{\Psi,J})$, we get

$$[w_{\lambda,Y}(y_1;1), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

Subcase 1b. $m_{\lambda,Y}(1)=0$. This case is, in fact, a special case of Subcase $1a$. Note that $w_{\lambda,Y}(y_1;1)=y_1$. Following the Subcase $1a$ step-by-step we have: Let $m_{\lambda,Y}(2)$ or $m_{\lambda,Y}(3)\geq 1$. If $m_{\lambda,Y}(5)$ or $m_{\lambda,Y}(6)\geq 1$, then the equations (16)-(21), and, by the construction of $L(\widetilde{W}_{\Psi,J})$ (It is an ideal in $L(W_\Psi)$.), the corresponding Lie commutator $[y_1; w(p_{m_{\lambda,Y}(2)}),\ldots,w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}$. Thus we assume that $m_{\lambda,Y}(5)=m_{\lambda,Y}(6)=0$. Let $m_{\lambda,Y}(4)\geq 1$. As in the Subcase $1a$, a similar analysis of $[y_1, w(p_{m_{\lambda,Y}(2)}), w(p_{m_{\lambda,Y}(3)})]$ (in the equations (16)-(21)), we obtain a $\mathbb{Z}$-submodule $W_{\lambda,Y}(y_1;m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)$ of $J_{1,\lambda}^{c+2}$. Furthermore, a $\mathbb{Z}$-submodule $W_{\lambda,Y}(y_1;4)$ of $J_{1,\lambda}^{c+2}$ is constructed.

Define

$$\begin{aligned} V_{1,\lambda}^{c+2} &= \textstyle\sum_{Y\in\mathcal{G}_\lambda}(W_{\lambda,Y}(y_1;m_{\lambda,Y}(1),m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)+W_{\lambda,Y}(y_1;1,4)+ \\ & \quad W_{\lambda,Y}(y_1;m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)+W_{\lambda,Y}(y_1;4)). \end{aligned}$$

We call the Lie commutators that generate the above $\mathbb{Z}$-modules and are coming from the equations $(C1)$ *the extended Lie commutators from the equations* $(C1)$. Each extended Lie commutator produces an element of $J\cap L_{\text{grad}}^{c+2}(W_\Psi^{(1)})$. Such an element is called *the corresponding extended element from the equations* $(C1)$. By the analysis in the Subcases $1a$ and $1b$, $V_{1,\lambda}^{c+2}$ is the direct sum of the aforementioned $\mathbb{Z}$-modules. Thus

$$\begin{aligned} V_{1,\lambda}^{c+2} &= \textstyle\bigoplus_{Y\in\mathcal{G}_\lambda}(W_{\lambda,Y}(y_1;m_{\lambda,Y}(1),m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)\oplus W_{\lambda,Y}(y_1;1,4)\oplus \\ & \quad W_{\lambda,Y}(y_1;m_{\lambda,Y}(2)+m_{\lambda,Y}(3),4)\oplus W_{\lambda,Y}(y_1;4)). \end{aligned}$$

Furthermore, it is clearly enough that $V_{1,\lambda}^{c+2} \cap J_C^{c+2} = \{0\}$, and $J_{1,\lambda}^{c+2} + J_{1,\lambda+1}^{c+2} + J_C^{c+2} = V_{1,\lambda}^{c+2} \oplus (J_{1,\lambda+1}^{c+2} + J_C^{c+2})$. By applying an inverse induction on the Part(c) (starting from $\lambda = (c)$), we have

$$J_1^{c+2} + J_C^{c+2} = U_1^{c+2} \oplus J_C^{c+2},$$

where $U_1^{c+2} = \bigoplus_{\lambda \in \mathrm{Part}(c)} V_{1,\lambda}^{c+2}$.

Case 2. An analysis of $J_{2,\lambda}^{c+2}$ modulo $(J_{2,\lambda+1}^{c+2} + J_C^{c+2})$. Recall that, for $\lambda \in \mathrm{Part}(c)$,

$$J_{2,\lambda}^{c+2} + J_C^{c+2} = [L^2(V_2); \Phi_\lambda/\Phi_{\lambda+1}] + J_{2,\lambda+1}^{c+2} + J_C^{c+2}.$$

By Corollary 3, we let $[L^2(V_2); \Phi_\lambda/\Phi_{\lambda+1}]$ be $\mathbb{Z}$-spanned by all Lie commutators of the form $[y_2; w(p_{m_{\lambda,Y}(2)}, p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})]$ with $Y \in \mathcal{G}_\lambda$. We separate two subcases, namely, $m_{\lambda,Y}(2) \geq 1$ and $m_{\lambda,Y}(2) = 0$. Instead of the equations $(C1)$, we use the equations $(C2)$. We point out that in order to construct similar expressions as in the Case 1 (the equations (16)-(21)), we apply the Jacobi identity several times. Following the analysis of case 1 step-by-step, we construct a $\mathbb{Z}$-submodule $V_{2,\lambda}^{c+2}$ of $J_{2,\lambda}^{c+2}$ such that $J_{2,\lambda}^{c+2} + J_{2,\lambda+1}^{c+2} + J_C^{c+2} = V_{2,\lambda}^{c+2} \oplus (J_{2,\lambda+1}^{c+2} + J_C^{c+2})$. In fact,

$$\begin{aligned} V_{2,\lambda}^{c+2} \;\; = \;\; & \textstyle\bigoplus_{Y \in \mathcal{G}_\lambda} (W_{\lambda,Y}(y_2; m_{\lambda,Y}(2), m_{\lambda,Y}(1) + m_{\lambda,Y}(3), 4) \oplus W_{\lambda,Y}(y_2; 2, 4) \oplus \\ & W_{\lambda,Y}(y_2; m_{\lambda,Y}(1) + m_{\lambda,Y}(3), 4) \oplus W_{\lambda,Y}(y_2; 4)). \end{aligned}$$

As in case 1, by applying an inverse induction on the Part(c) (starting from $\lambda = (c)$), we have

$$J_2^{c+2} + J_C^{c+2} = U_2^{c+2} \oplus J_C^{c+2},$$

where $U_2^{c+2} = \bigoplus_{\lambda \in \mathrm{Part}(c)} V_{2,\lambda}^{c+2}$.

Case 3. An analysis of $J_{3,\lambda}^{c+2}$ modulo $(J_{3,\lambda+1}^{c+2} + J_C^{c+2})$. Recall that, for $\lambda \in \mathrm{Part}(c)$,

$$J_{3,\lambda}^{c+2} + J_C^{c+2} = [L^2(V_3); \Phi_\lambda/\Phi_{\lambda+1}] + J_{3,\lambda+1}^{c+2} + J_C^{c+2}.$$

By Corollary 3, we let $[L^2(V_3); \Phi_\lambda/\Phi_{\lambda+1}]$ be $\mathbb{Z}$-spanned by all Lie commutators of the form $[y_3; w(p_{m_{\lambda,Y}(3)}, p_{m_{\lambda,Y}(1)}, \ldots, p_{m_{\lambda,Y}(6)})]$ with $Y \in \mathcal{G}_\lambda$. We separate two subcases, namely, $m_{\lambda,Y}(3) \geq 1$ and $m_{\lambda,Y}(3) = 0$. Instead of the equations $(C1)$, we use the equations $(C3)$. We point out that in order to construct similar expressions as in the Case 1 (the equations (16)-(21)), we apply the Jacobi identity several times. Following the analysis of case 1 step-by-step, we construct a $\mathbb{Z}$-submodule $V_{3,\lambda}^{c+2}$ of $J_{3,\lambda}^{c+2}$ such that $J_{3,\lambda}^{c+2} + J_{3,\lambda+1}^{c+2} + J_C^{c+2} = V_{3,\lambda}^{c+2} \oplus (J_{3,\lambda+1}^{c+2} + J_C^{c+2})$. In fact,

$$\begin{aligned} V_{3,\lambda}^{c+2} \;\; = \;\; & \textstyle\bigoplus_{Y \in \mathcal{G}_\lambda} (W_{\lambda,Y}(y_3; m_{\lambda,Y}(3), m_{\lambda,Y}(1) + m_{\lambda,Y}(2), 4) \oplus W_{\lambda,Y}(y_3; 3, 4) \oplus \\ & W_{\lambda,Y}(y_3; m_{\lambda,Y}(1) + m_{\lambda,Y}(2), 4) \oplus W_{\lambda,Y}(y_3; 4)). \end{aligned}$$

As in case 1, by applying an inverse induction on the Part(c) (starting from $\lambda = (c)$), we have

$$J_3^{c+2} + J_C^{c+2} = U_3^{c+2} \oplus J_C^{c+2},$$

where $U_3^{c+2} = \bigoplus_{\lambda \in \mathrm{Part}(c)} V_{3,\lambda}^{c+2}$.

Case 4. An analysis of $J^{c+2}_{\Psi,\lambda}$ modulo $(J^{c+2}_{\Psi,\lambda+1}+J^{c+2}_{C})$. Write $\mathcal{V}^{(2)}_1=\{\psi_2(y_6),\psi_2(y_7),\psi_2(y_9),\psi_2(y_{10})\}$ and $\mathcal{V}^{(2)}_2=\{\psi_2(y_{12}),\psi_2(y_{15})\}$. That is, $\mathcal{V}=\mathcal{V}^{(2)}_1\cup\mathcal{V}^{(2)}_2$ is a $\mathbb{Z}$-basis for $W^{(2)}_{2,\Psi}$. Recall that $[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}]$ is the $\mathbb{Z}$-submodule of $[J^2;T^c]$ spanned by all Lie commutators of the form $[v;w(p_{m_{\lambda,Y}(1)},\ldots,p_{m_{\lambda,Y}(6)})]$ with $v\in\mathcal{V}$ and $Y\in\mathcal{G}_\lambda$. Thus

$$[W^{(2)}_{2,\Psi};\Phi_\lambda/\Phi_{\lambda+1}]=[[V_2,V_1]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]+[[V_3,V_1]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]+[[V_3,V_2]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}].$$

By the definition of J^{c+2}_C, we have

$$[[V_2,V_1]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]+[[V_3,V_1]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]\subseteq J^{c+2}_C.$$

Thus

$$J^{c+2}_{\Psi,\lambda}+J^{c+2}_{\Psi,\lambda+1}+J^{c+2}_C=[[V_3,V_2]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]+J^{c+2}_{\Psi,\lambda+1}+J^{c+2}_C.$$

If $m_{\lambda,Y}(1)=0$, then, by the definition of J^{c+2}_C, each Lie commutator

$$[v;w(p_{m_{\lambda,Y}(2)},\ldots,p_{m_{\lambda,Y}(6)})]\in J^{c+2}_C$$

for all $v\in\mathcal{V}^{(2)}_2$. Thus we assume that $m_{\lambda,Y}(1)\geq 1$. If $m_{\lambda,Y}(5)$ or $m_{\lambda,Y}(6)\geq 1$, then, by the equations $(C3)-(C5)$, we have

$$[v,w(p_{m_{\lambda,Y}(1)}),\ldots,w(p_{m_{\lambda,Y}(6)})]\in J^{c+2}_C.$$

Thus we assume that $m_{\lambda,Y}(5)=m_{\lambda,Y}(6)=0$. For $v\in\mathcal{V}^{(2)}_2$, let $\langle v\rangle$ denote the cyclic $\mathbb{Z}$-module generated by v. Since $[V_3,V_2]^{(2)}=\bigoplus_{v\in\mathcal{V}^{(2)}_2}\langle v\rangle$, we have

$$[[V_3,V_2]^{(2)};\Phi_\lambda/\Phi_{\lambda+1}]=\sum_{v\in\mathcal{V}^{(2)}_2}[\langle v\rangle;\Phi_\lambda/\Phi_{\lambda+1}]. \tag{22}$$

First we shall work with $v=\psi_2(y_{12})$. Similar arguments may be applied to $v=\psi_2(y_{15})$. Suppose that $w_{1,1}=x_2$. Then $w_\ell(p_{m_{\lambda,Y}(1)})=x_2^{\mu_{1,1}}$ and $w(p_{m_{\lambda,Y}(1)})=x_2^{\mu_{1,1}}$. By the equations $(C4)$, the Jacobi identity and the equations (D), we have

$$\begin{array}{rcl}[\psi_2(y_{12});x_2^{\mu_{1,1}}] &=& [\psi_2(y_{10}),\ _{(\mu_{1,1}-1)}x_2,x_3]-[y_8,\ _{\mu_{1,1}}x_2,x_3]-[y_5,\ _{(\mu_{1,1}-1)}x_2,x_5]-\\ && [\psi_2(y_6),\ _{(\mu_{1,1}-1)}x_2,x_5]+[\psi_2(y_7),\ _{(\mu_{1,1}-1)}x_2,x_5]+[y_{11}\ _{(\mu_{1,1}-1)}x_2,x_3]-\\ && [\psi_2(y_6),\ _{(\mu_{1,1}-1)}x_2,x_6]+[\psi_2(y_7),\ _{(\mu_{1,1}-1)}x_2,x_6]-\\ && [y_5,\ _{(\mu_{1,1}-1)}x_2,x_6]+w_{(12,2)}+w_{(12,2,J)}, \qquad (23)\end{array}$$

where $w_{(12,2)}\in L^{2+m_{\lambda,Y}(1)}_{\mathrm{grad,e}}(W^{(1)}_\Psi)$, with $e\geq 2$, and $w_{(12,2,J)}\in J_C$. Thus we assume that $w_{1,1}\neq x_2$. By using similar arguments as before, we have

$$\begin{array}{rcl}[\psi_2(y_{12});w(p_{m_{\lambda,Y}(1)})] &=& f(\psi_2(y_9);w'_{1,1}w_{1,1}^{(\mu_{1,1}-1)}\cdots w_{\tau(1),1}^{\mu_{\tau(1),1}}x_3)-\\ && f(y_4;w'_{1,1}w_{1,1}^{(\mu_{1,1}-1)}\cdots w_{\tau(1),1}^{\mu_{\tau(1),1}}x_5)+\\ && f(y_8;w'_{1,1}w_{1,1}^{(\mu_{1,1}-1)}\cdots w_{\tau(1),1}^{\mu_{\tau(1),1}}x_3)+\\ && f(y_4;w'_{1,1}w_{1,1}^{(\mu_{1,1}-1)}\cdots w_{\tau(1),1}^{\mu_{\tau(1),1}}x_6)+\\ && w_{(12,m_{\lambda,Y}(1))}+w_{(12,m_{\lambda,Y}(1),J)}, \qquad (24)\end{array}$$

where $w'_{1,1}$ is the unique element in $\mathcal{V}_1^*$ (the free monoid on $\mathcal{V}_1$) such that $w_{1,1} = x_1 w'_{1,1}$, $w_{(12,m_{\lambda,Y}(1))} \in L_{\mathrm{grad,e}}^{2+m_{\lambda,Y}(1)}(W_\Psi^{(1)})$ (with $e \geq 2$) and $w_{(12,m_{\lambda,Y}(1),J)} \in J_C$. By the equation (15) (for $t = 2,3$), the equation (23), the Jacobi identity and the equations (D), we get

$$\begin{aligned}
[[\psi_2(y_{12}); x_2^{\mu_{1,1}}]; w(p_{m_{\lambda,Y}(2)}), w(p_{m_{\lambda,Y}(3)})] = \; & f([\psi_2(y_{10}), x_2]; x_2^{\mu_{1,1}-2} x_3 w_\ell(p_{m_{\lambda,Y}(2)}), w_\ell(p_{m_{\lambda,Y}(3)})) - \\
& f([y_8, x_2]; x_2^{\mu_{1,1}-1} x_3 w_\ell(p_{m_{\lambda,Y}(2)}), w_\ell(p_{m_{\lambda,Y}(3)})) - \\
& f([y_5, x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_5 w_\ell(p_{m_{\lambda,Y}(3)})) - \\
& f([\psi_2(y_6), x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_5 w_\ell(p_{m_{\lambda,Y}(3)})) + \\
& f([\psi_2(y_7), x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_5 w_\ell(p_{m_{\lambda,Y}(3)})) + \\
& f([y_{11}, x_2]; x_2^{\mu_{1,1}-1} x_3 w_\ell(p_{m_{\lambda,Y}(2)}), w_\ell(p_{m_{\lambda,Y}(3)})) - \\
& f([\psi_2(y_6), x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_6 w_\ell(p_{m_{\lambda,Y}(3)})) + \\
& f([\psi_2(y_7), x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_6 w_\ell(p_{m_{\lambda,Y}(3)})) - \\
& f([y_5, x_2]; x_2^{\mu_{1,1}-2} w_\ell(p_{m_{\lambda,Y}(2)}), x_6 w_\ell(p_{m_{\lambda,Y}(3)})) + \\
& w_{(12,1)} + w_{(12,1,J)}, \qquad (25)
\end{aligned}$$

where $w_{(12,1)} \in L_{\mathrm{grad,e}}^{2+\mu_{1,1}+m_{\lambda,Y}(2)+m_{\lambda,Y}(3)}(W_\Psi^{(1)})$, $e \geq 2$, and $w_{(12,1,J)} \in J_C$. By applying similar arguments as above and using the equation (24), we obtain a similar expression for $[[\psi_2(y_{12}); w(p_{m_{\lambda,Y}(1)})]; w(p_{m_{\lambda,Y}(2)}), w(p_{m_{\lambda,Y}(3)})]$ (when $w_{1,1} \neq x_2$).

For the next few lines, we write

$$w_{(1,2,3)}^{(12)} = [[\psi_2(y_{12}); w(p_{m_{\lambda,Y}(1)})]; w(p_{m_{\lambda,Y}(2)}), w(p_{m_{\lambda,Y}(3)})].$$

For $m_{\lambda,Y}(4) = 0$, let $W_{\lambda,Y,\Psi}^{(12)}(m_{\lambda,Y}(1), m_{\lambda,Y}(2), m_{\lambda,Y}(3))$ be the $\mathbb{Z}$-submodule of $J_{\Psi,\lambda}^{c+2}$ spanned by all Lie commutators $w_{(1,2,3)}^{(12)}$. Finally, we assume that $m_{\lambda,Y}(4) \geq 1$. The analysis of $w_{(1,2,3)}^{(12)}$ allows us to assume that it is written as $-v_1 \pm v_2 + v_3 + \alpha v_4 + \widetilde{w}_{(1,2,3)}^{(12)}$ with $\alpha \in \{-1, 0\}$, $\widetilde{w}_{(1,2,3)}^{(12)} \in L_{\mathrm{grad}}^{c+2}(W_\Psi)$, $v_1, v_2, v_3, v_4 \in \mathcal{W}_\Psi^{(1)}$ and v_1, v_2, v_3, v_4 do not occur in the expression of $\widetilde{w}_{(1,2,3)}^{(12)}$. We proceed as in Subcase 1a. Then either $v_1 = \zeta_{1,4}$ or $v_2 = \zeta_{1,4}$ or $v_3 = \zeta_{1,4}$ or $v_4 = \zeta_{1,4}$ or $v_1, v_2, v_3, v_4 \neq \zeta_{1,4}$. Let us assume that $v_1, v_2, v_3, v_4 \neq \zeta_{1,4}$. Then, by Lemma 19, different Lyndon words occur in the expressions of $[v_1; w(p_{m_{\lambda,Y}(4)})]$, $[v_2; w(p_{m_{\lambda,Y}(4)})]$, $[v_3; w(p_{m_{\lambda,Y}(4)})]$ and $[v_4; w(p_{m_{\lambda,Y}(4)})]$. We replace the (unique) Lyndon polynomial corresponding to the smallest Lyndon word (by means of the ordering of $\mathcal{W}_\Psi^{(1)}$) by $[w_{(1,2,3)}^{(12)}; w(p_{m_{\lambda,Y}(4)})]$. We point out the above (different) Lyndon words do not appear in the expression of $[\widetilde{w}_{(1,2,3)}^{(12)}; w(p_{m_{\lambda,Y}(4)})]$. Similar arguments may be applied if $v_1 = \zeta_{1,4}$ or $v_2 = \zeta_{1,4}$, $v_3 = \zeta_{1,4}$ or $v_4 = \zeta_{1,4}$. Furthermore, we deal with similar arguments the cases either $m_{\lambda,Y}(2) \geq 1$ or $m_{\lambda,Y}(3) \geq 1$. We write $W_{\lambda,Y,\Psi}^{(12)}(1, m_{\lambda,Y}(2) + m_{\lambda,Y}(3), 4)$ for the $\mathbb{Z}$-submodule of $J_{\Psi,\lambda}^{c+2}$ spanned by all Lie commutators of the form $[w_{(1,2,3)}^{(12)}; w(p_{m_{\lambda,Y}(4)})]$ mentioned above with $m_{\lambda,Y}(1) \geq 1$, $m_{\lambda,Y}(2) + m_{\lambda,Y}(3) \geq 1$ and $m_{\lambda,Y}(4)$.

Next we assume that $m_{\lambda,Y}(2) = m_{\lambda,Y}(3) = 0$. Note that

$$[\psi_2(y_{12}), w(p_{m_{\lambda,Y}(1)}), w(p_{m_{\lambda,Y}(4)})] \in L_{\mathrm{grad}}(\widetilde{W}_{\Psi,J})$$

and so, if $m_{\lambda,Y}(5) + m_{\lambda,Y}(6) \geq 0$, we have

$$[\psi_2(y_{12}), w(p_{m_{\lambda,Y}(1)}), w(p_{m_{\lambda,Y}(4)}), w(p_{m_{\lambda,Y}(5)}), w(p_{m_{\lambda,Y}(6)})] \in J_C^{c+2}.$$

Define

$$V^{c+2}_{12,\Psi,\lambda} = \sum_{Y\in\mathcal{G}_\lambda} (W^{(12)}_{\lambda,Y,\Psi}(m_{\lambda,Y}(1), m_{\lambda,Y}(2), m_{\lambda,Y}(3)) + W^{(12)}_{\lambda,Y,\Psi}(1, m_{\lambda,Y}(2) + m_{\lambda,Y}(3), 4)).$$

By the above analysis, $V^{c+2}_{12,\Psi,\lambda}$ is the direct sum of the aforementioned $\mathbb{Z}$-modules. Thus

$$V^{c+2}_{12,\Psi,\lambda} = \bigoplus_{Y\in\mathcal{G}_\lambda} (W^{(12)}_{\lambda,Y,\Psi}(m_{\lambda,Y}(1), m_{\lambda,Y}(2), m_{\lambda,Y}(3)) \oplus W^{(12)}_{\lambda,Y,\Psi}(1, m_{\lambda,Y}(2) + m_{\lambda,Y}(3), 4)).$$

It is clearly enough that $V^{c+2}_{12,\Psi,\lambda} \cap J^{c+2}_C = \{0\}$. Following the same method (and giving the same arguments) as above for the case $v = \psi_2(y_{15})$, and using the analogous equations, we have

$$V^{c+2}_{15,\Psi,\lambda} = \bigoplus_{Y\in\mathcal{G}_\lambda} (W^{(15)}_{\lambda,Y,\Psi}(m_{\lambda,Y}(1), m_{\lambda,Y}(2), m_{\lambda,Y}(3)) \oplus W^{(15)}_{\lambda,Y,\Psi}(1, m_{\lambda,Y}(2) + m_{\lambda,Y}(3), 4)).$$

Having in mind that the elements in the equations $(C3)-(C5)$ are $\mathbb{Z}$-linearly independent, we may show that the above (in all possible cases) extended Lie commutators from the equations $(C3)-(C6)$ are $\mathbb{Z}$-linearly independent. This shows that the sum $\sum_{Y\in\mathcal{G}_\lambda}(V^{c+2}_{12,\Psi,\lambda} + V^{c+2}_{15,\Psi,\lambda})$ is direct. Define

$$V^{c+2}_{\Psi,\lambda} = \bigoplus_{Y\in\mathcal{G}_\lambda} (V^{c+2}_{12,\Psi,\lambda} \oplus V^{c+2}_{15,\Psi,\lambda}).$$

Since $V^{c+2}_{\Psi,\lambda} \cap J^{c+2}_C = \{0\}$, we get from the equation (22) and the above analysis,

$$J^{c+2}_{\Psi,\lambda} + J^{c+2}_{\Psi,\lambda+1} + J^{c+2}_C = V^{c+2}_{\Psi,\lambda} \oplus (J^{c+2}_{\Psi,\lambda+1} + J^{c+2}_C).$$

By applying an inverse induction on the $\mathrm{Part}(c)$ (starting from $\lambda = (c)$), we have

$$J^{c+2}_\Psi + J^{c+2}_C = U^{c+2}_\Psi \oplus J^{c+2}_C,$$

where $U^{c+2}_\Psi = \bigoplus_{\lambda\in\mathrm{Part}(c)} V^{c+2}_{\Psi,\lambda}$.

Write $U^{c+2} = U^{c+2}_1 + U^{c+2}_2 + U^{c+2}_3 + U^{c+2}_\Psi$. Let $\mathcal{E}_{c+2}$ be the set consisting of the extended Lie commutators (in all possible cases) which create $U^{c+2}_1, U^{c+2}_2, U^{c+2}_3, U^{c+2}_\Psi$, respectively, and the Lie commutators which are not coming from the equations $(C1)-(C5)$. By the cases 1, 2 and 3, the ordering on $\mathcal{W}^{(1)}_\Psi$ and having in mind that the elements in the equations $(C1)-(C5)$ are $\mathbb{Z}$-linearly independent, we may show that $\mathcal{E}_{c+2}$ is $\mathbb{Z}$-linearly independent. So, U^{c+2} is a direct sum. But

$$\begin{aligned} J^{c+2} &= J^{c+2}_1 + J^{c+2}_2 + J^{c+2}_3 + J^{c+2}_\Psi \\ &= U^{c+2} + J^{c+2}_C. \end{aligned}$$

Since $U^{c+2} \cap J^{c+2}_C = \{0\}$, we have

$$J^{c+2} = U^{c+2} \oplus J^{c+2}_C. \tag{26}$$

We claim that

$$J^{c+2} = (J \cap L^{c+2}_{\mathrm{grad}}(W^{(1)}_\Psi)) \oplus J^{c+2}_C. \tag{27}$$

¿From the analysis of the Lie commutators in the cases 1, 2, 3 and 4, it is clearly enough that

$$U^{c+2} \subseteq (J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})) \oplus J^{c+2}_C.$$

and so, we obtain the desired $\mathbb{Z}$-module decomposition of J^{c+2} in the equation (27). From the equations (26) and (27), we get $J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}) \cong U^{c+2}$ as $\mathbb{Z}$-modules. From the analysis of cases 1, 2, 3 and 4, we have the corresponding extended elements from the equations $(C1)-(C5)$ and the Lie commutators which are not coming from $(C1)-(C5)$ belong to $J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})$. Write $\mathcal{E}^{\rm corr}_{c+2}$ for the set of all aforementioned Lie elements which belong to $J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})$. Since $J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}) \cong U^{c+2}$ as (torsion-free) $\mathbb{Z}$-modules and $\mathcal{E}^{\rm corr}_{c+2}$ is $\mathbb{Z}$-linearly independent, we have $J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})$ is $\mathbb{Z}$-spanned by $\mathcal{E}^{\rm corr}_{c+2}$. By the form of the elements of $\mathcal{E}^{\rm corr}_{c+2}$, it is easy to construct a set consisting of Lie polynomials in $L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})$, and let $(L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}))^*$ denote the $\mathbb{Z}$-module spanned by this set, such that

$$L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}) = (J \cap L^{c+2}_{\rm grad}(W^{(1)}_{\Psi})) \oplus (L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}))^* \tag{28}$$

Since

$$L^{c+2} = L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}) \oplus J^{c+2}_C,$$

we obtain from the equations (28) and (27),

$$L^{c+2} = (L^{c+2}_{\rm grad}(W^{(1)}_{\Psi}))^* \oplus J^{c+2}.$$

Corollary 4 *L/J is torsion-free as $\mathbb{Z}$-module. Furthermore, J is a free Lie algebra.*

Proof. By the above theorem, for $c \geq 2$, L^{c+2}/J^{c+2} is torsion free $\mathbb{Z}$-module. Since $J = \bigoplus_{c\geq 0} J^{c+2}$ and $L/J \cong \bigoplus_{c\geq 0}(L^{c+2}/J^{c+2})$ as $\mathbb{Z}$-modules, we have L/J is torsion-free. By a result of Witt (see [2, Theorem 2.4.2.5]), we have J is a free Lie algebra.

6 The Lie algebra of M_3

In this section, we deduce a presentation of $\text{Ł}(M_3)$. In particular, we prove in Theorem 3 that $L/J \cong \text{Ł}(M_3)$ as Lie algebras. Recall that F denotes a free group of rank 6 with a free generating set $\{a_1, \ldots, a_6\}$ with ordering $a_1 < a_2 < \cdots < a_6$. Furthermore, $\mathcal{R}_{\mathcal{V}} = \{r_1, \ldots, r_9\}$ and $M_3 = F/N$ where N is generated as a group by the set $\{r^g = g^{-1}rg : r \in \mathcal{R}_{\mathcal{V}}, g \in F\}$. Write $\widetilde{\mathcal{R}_{\mathcal{V}}} = \{(r^{\pm 1}, g) : r \in \mathcal{R}_{\mathcal{V}}, g \in F \setminus \{1\}\}$. Since $N\gamma_3(F)/\gamma_3(F)$ is generated by the set $\{r\gamma_3(F) : r \in \mathcal{R}_{\mathcal{V}}\}$, $N\gamma_3(F)/\gamma_3(F) = J^2$ and since $L^2 = (L^2)^* \oplus J^2$ as $\mathbb{Z}$-modules, where $(L^2)^* = W^{(1)}_{2,\Psi}$ (by Lemma 16 (I)), we have the set $\{r\gamma_3(F) : r \in \mathcal{R}_{\mathcal{V}}\}$ is $\mathbb{Z}$-linearly independent. Hence $\mathcal{R}_{\mathcal{V}} \cap \widetilde{\mathcal{R}_{\mathcal{V}}} = \emptyset$ and so, N is generated by the disjoint union $\mathcal{R}_{\mathcal{V}} \cup \widetilde{\mathcal{R}_{\mathcal{V}}}$. Write $g = a^{\pm 1}_{i_1} \cdots a^{\pm 1}_{i_\mu}$ as a reduced word in F. Since $(a, bc) = (a, c)(a, b)(a, b, c)$, we may write $(r^{\pm 1}, g)$ as a product of group commutators of the form $(r^{\pm 1}, a^{\pm 1}_{j_1}, \ldots, a^{\pm 1}_{j_s})$ with $s \geq 1$ and $j_1, \ldots, j_s \in \{1, \ldots, 6\}$. Writing

$$\mathcal{R}'_{\mathcal{V}} = \{(r^{\pm 1}, a^{\pm 1}_{j_1}, \ldots, a^{\pm 1}_{j_s}), r \in \mathcal{R}_{\mathcal{V}}, s \geq 1, j_1, \ldots, j_s \in \{1, \ldots, 6\}\},$$

we have N is generated by the disjoint union $\mathcal{R}_{\mathcal{V}} \cup \mathcal{R}'_{\mathcal{V}}$. Since N is a non-trivial normal subgroup of F and the index of N in F is not finite (since $N \subseteq F'$), we have by a result of Nielsen-Schreier that N is a free group of infinite rank (see, also, [15, Proposition 3.12]).

For a positive integer d, let $N_d = N \cap \gamma_d(F)$. Note that for $d \leq 2$, we have $N_d = N$. Also, for $d \geq 2$, $N_{d+1} = N_d \cap \gamma_{d+1}(F)$. Since $(g_1, \ldots, g_\kappa)^f = (g_1^f, \ldots, g_\kappa^f)$ for all $f \in F$, and N is normal, we obtain $\{N_d\}_{d\geq 2}$ is a normal (descending) series of N. Clearly each N_d is normal in F. Since $(N_\kappa, N_\ell) \subseteq N_{\kappa+\ell}$ for all $\kappa, \ell \geq 2$, we have $\{N_d\}_{d\geq 2}$ is a central series of N. Define $\mathcal{I}_d(N) = N_d\gamma_{d+1}(F)/\gamma_{d+1}(F)$. It is easily verified that $\mathcal{I}_d(N) \cong N_d/N_{d+1}$ as $\mathbb{Z}$-modules. By our definitions, identifications and the above discussion, $\mathcal{I}_2(N) = N\gamma_3(F)/\gamma_3(F) = J^2$. Let $\mathcal{B}_2^*$, $\mathcal{B}_{2,J}$ be the natural $\mathbb{Z}$-bases for $(L^2)^*$, J^2, respectively. By Lemma 16 (I), we have the union $\mathcal{B}_2^* \cup \mathcal{B}_{2,J}$ is disjoint, and it is a $\mathbb{Z}$-basis of L^2. Note that we may consider $\mathcal{B}_2^* = \mathcal{V}^*$ and $\mathcal{B}_{2,J} = \mathcal{V}$. Since $\mathcal{R}_\mathcal{V}\gamma_3(F)/\gamma_3(F) \subseteq \mathcal{I}_2(N)$ and $\mathcal{V}$ is a $\mathbb{Z}$-basis of J^2, we have $\mathcal{R}_\mathcal{V}\gamma_3(F)/\gamma_3(F)$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{I}_2(N)$. Thus

$$r_1^{m_1} \cdots r_9^{m_9} \notin \gamma_3(F) \tag{34}$$

for any $m_1, \ldots, m_9 \in \mathbb{Z}$ with $m_1 + \cdots + m_9 \neq 0$. Write N_{21} and N_{22} for the subgroups of N generated by $\mathcal{R}_\mathcal{V}$ and $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}_\mathcal{V}, s \geq 1, g_1, \ldots, g_s \in F \setminus \{1\}\}$, respectively. Since $(ab, c) = (a, c)(a, c, b)(b, c)$ for all $a, b, c \in F$, we have N_{22} is normal in N. Since $N = N_{21}N_{22}$ and $N_{22} \subseteq \gamma_3(F)$, we have, by the modular law,

$$N_3 = N \cap \gamma_3(F) = (N_{21} \cap \gamma_3(F))N_{22}.$$

By (34), we get $N_{21} \cap \gamma_3(F)$ is generated by elements of length at least 4 and so, $N_{21} \cap \gamma_3(F) \subseteq N_{22}$. Thus N_3 is generated by the set $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}_\mathcal{V}, s \geq 1, g_1, \ldots, g_s \in F \setminus \{1\}\}$. By our definitions and identifications, $\mathcal{I}_3(N) = N_3\gamma_4(F)/\gamma_4(F) = J^3$. Write $(L^3)^* = (W_{3,\Psi}^{(1)})^*$. By Lemma 16 (II)

$$J^3 = \Big(\bigoplus_{i=2}^{3}[L^2(V_1), V_i]\Big) \oplus \Big(\bigoplus_{\substack{i=1\\ i\neq 2}}^{3}[L^2(V_2), V_i]\Big) \oplus \Big(\bigoplus_{i=1}^{2}[L^2(V_3), V_i]\Big) \oplus [[V_3, V_2]^{(2)}, V_1] \oplus J_C^3.$$

Let $\mathcal{B}_3^*$, $\mathcal{B}_{3,J}$ be the natural $\mathbb{Z}$-bases for $(L^3)^*$, J^3, respectively. By Lemma 16 (III), we have the union $\mathcal{B}_3^* \cup \mathcal{B}_{3,J}$ is disjoint, and it is a $\mathbb{Z}$-basis of L^3. Hence we choose a $\mathbb{Z}$-basis $\mathcal{B}_{3,J} = \{b_1^{(3)}, \ldots, b_{m(3)}^{(3)}\}$, with $m(3) = \mathrm{rank}(J^3)$, of J^3 consisting of Lie commutators of length 3 where at least one element of $\mathcal{V}$ occurs in each Lie commutator of the basis elements. So, each basis element of J^3 is written as a $\mathbb{Z}$-linear combination of Lie commutators of degree 3 of the above form $[\ell_1, \ell_2, \ell_3]$. By replacing the elements of $\mathcal{V}$ occurring in $b_\kappa^{(3)}$ by elements of $\mathcal{R}_\mathcal{V}$, using χ, we may view each $b_\kappa^{(3)}$ as an element of N_3. We write $\mathcal{B}_{3,J,f}$ for the set $\mathcal{B}_{3,J}$ viewed its elements as elements of N_3. (For example, $[x_2, x_1, x_1]$ is replaced by (r_1, a_1) or, $[\psi_2(y_{15}), x_2] - [\psi_2(y_{10}), x_4] + [\psi_2(y_6), x_2]$ is replaced by $(r_5, a_2)(r_4, a_4)^{-1}(r_6, a_2)$.) Since $\mathcal{B}_{3,J,f}\gamma_4(F)/\gamma_4(F) \subseteq \mathcal{I}_3(N) = J^3$ and $\mathcal{B}_{3,J}$ is a $\mathbb{Z}$-basis of J^3, we have $\mathcal{B}_{3,J,f}\gamma_4(F)/\gamma_4(F)$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{I}_3(N)$. Thus

$$(b_1^{(3)})^{n_1} \cdots (b_{m(3)}^{(3)})^{n_{m(3)}} \notin \gamma_4(F) \tag{35}$$

for any $n_1, \ldots, n_{m(3)} \in \mathbb{Z}$ with $n_1 + \cdots + n_{m(3)} \neq 0$. Note that the set $\mathcal{B}_{3,J,f}\gamma_4(F)/\gamma_4(F)$ is a $\mathbb{Z}$-basis of $\mathcal{I}_3(N)$. By (35), $\mathcal{B}_{3,J,f} \cap \gamma_4(F) = \emptyset$. Write N_{31} and N_{32} for the subgroups of N_3 generated by the sets $\mathcal{B}_{3,J,f}$ and $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}_\mathcal{V}, s \geq 2, g_1, \ldots, g_s \in F \setminus \{1\}\}$, respectively. Note that N_{32} is normal in N_3 and so, $N_{31}N_{32}$ is a subgroup of N_3. We claim that $N_3 = N_{31}N_{32}$. It is enough to show that $N_3 \subseteq N_{31}N_{32}$. Furthermore, it is enough to show that

$(r^{\pm 1}, a_i) \in N_{31}N_{32}$ for $r \in \mathcal{R}_{\mathcal{V}}$, $i = 1, \ldots, 6$. Since $(r^{-1}, a_i) = (r, a_i)^{-1}((r, a_i)^{-1}, r^{-1}) \in N_3$ and N_3 is normal in F, it is enough to show that $(r, a_i) \in N_{31}N_{32}$ for $r \in \mathcal{R}_{\mathcal{V}}$, $i = 1, \ldots, 6$. For $i = 1, \ldots, 6$ and $r \in \mathcal{R}_{\mathcal{V}}$, there are unique $n_{1i}, \ldots, n_{m(3)i} \in \mathbb{Z}$ such that

$$(r, a_i)\gamma_4(F) = (b_1^{(3)})^{n_{1i}} \cdots (b_{m(3)}^{(3)})^{n_{m(3)i}}\gamma_4(F).$$

In fact, working the elements $b_1^{(3)}, \ldots, b_{m(3)}^{(3)}$, appearing in the above product, in $N_{31}N_{32} \subseteq \gamma_3(F)$ (Since the elements $b_1^{(3)}, \ldots, b_{m(3)}^{(3)}$ are regarded in N_3.) and using the identity $ab = ba(a, b)$, we form the product $(b_1^{(3)})^{n_{1i}} \cdots (b_{m(3)}^{(3)})^{n_{m(3)i}}$ in such a way

$$(r, a_i) = (b_1^{(3)})^{n_{1i}} \cdots (b_{m(3)}^{(3)})^{n_{m(3)i}} v$$

with $v \in \gamma_4(F)$. Following the procedure, it is clearly enough that $v \in N_{31}N_{32}$. Thus $(r, a_i) \in N_{31}N_{32}$ for all $r \in \mathcal{R}_{\mathcal{V}}$ and $i \in \{1, \ldots, 6\}$. Therefore $N_3 = N_{31}N_{32}$. Since $N_3 = N_{31}N_{32}$ and $N_{32} \subseteq \gamma_4(F)$, we have, by the modular law,

$$N_4 = N_3 \cap \gamma_4(F) = N_{31}N_{32} \cap \gamma_4(F) = (N_{31} \cap \gamma_4(F))N_{32}.$$

By (35), we get $N_{31} \cap \gamma_4(F)$ is generated by elements of length at least 6. Having in mind the way in which $b_1^{(3)}, \ldots, b_{m(3)}^{(3)}$ are regarded elements in N_3 and the definition of N_{32}, we have $N_{31} \cap \gamma_4(F) \subseteq N_{32}$. Thus N_4 is generated by the set $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}, s \geq 2, g_1, \ldots, g_s \in F \setminus \{1\}\}$ and so, $\mathcal{I}_4(N) = J^4$.

Proposition 2 *For a positive integer c, N_{c+2} is generated by the set $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}, s \geq c, g_1, \ldots, g_s \in F \setminus \{1\}\}$. Furthermore, $\mathcal{I}_{c+2}(N) = J^{c+2}$ for all $c \geq 1$.*

Proof. We proceed by induction on c, with $c \geq 1$. We have already shown our claim for $c = 1, 2$. Thus we assume that N_{c+2}, with $c \geq 2$, is generated by the set $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}, s \geq c, g_1, \ldots, g_s \in F \setminus \{1\}\}$ and so, by our definitions and identifications, $\mathcal{I}_{c+2}(N) = J^{c+2}$. By the proof of Theorem 2, there exists a $\mathbb{Z}$-module $(L^{c+2})^*$, say, such that $L^{c+2} = (L^{c+2})^* \oplus J^{c+2}$ as $\mathbb{Z}$-modules. By the equation (3) and Theorem 2, we have, for $c \geq 2$,

$$J^{c+2} = U_1^{c+2} \oplus U_2^{c+2} \oplus U_3^{c+2} \oplus U_\Psi^{c+2} \oplus L^{c+2}(V_1) \oplus L^{c+2}(V_2) \oplus L^{c+2}(V_3) \oplus L_{\text{grad}}^{c+2}(W_\Psi^{(2)}) \oplus L_{\text{grad}}^{c+2}(\widetilde{W}_{\Psi,J}).$$

Let $\mathcal{B}_{c+2}^*, \mathcal{B}_{c+2,J}$ be $\mathbb{Z}$-bases of $(L^{c+2})^*, J^{c+2}$, respectively. (By using the proof of Theorem 2, and the chosen $\mathbb{Z}$-bases of $L^{c+2}(V_1), L^{c+2}(V_2), L_{\text{grad}}^{c+2}(W_\Psi^{(2)})$ and $L_{\text{grad}}^{c+2}(\widetilde{W}_{\Psi,J})$, respectively.) Thus we have the union $\mathcal{B}_{c+2}^* \cup \mathcal{B}_{c+2,J}$ is disjoint, and it is a $\mathbb{Z}$-basis of L^{c+2}. Hence we choose a $\mathbb{Z}$-basis $\mathcal{B}_{c+2,J} = \{b_1^{(c+2)}, \ldots, b_{m(c+2)}^{(c+2)}\}$, with $m(c+2) = \text{rank}(J^{c+2})$, of J^{c+2} consisting of Lie commutators of length $c + 2$ where at least one element of $\mathcal{V}$ occurs in each Lie commutator of the basis elements. So, each basis element of J^{c+2} is written as a $\mathbb{Z}$-linear combination of Lie commutators of length $c+2$ of the aforementioned form $[\ell_1, \ldots, \ell_{c+2}]$. By replacing (as in the case $c = 2$) the elements of $\mathcal{V}$ occurring in $b_\kappa^{(c+2)}$ by the corresponding elements of $\mathcal{R}_{\mathcal{V}}$, we may view each $b_\kappa^{(c+2)}$ as an element of N_{c+2}. We write $\mathcal{B}_{c+2,J,f}$ for the set $\mathcal{B}_{c+2,J}$ viewed its elements as elements of N_{c+2}. Since $\mathcal{B}_{c+2,J,f}\gamma_{c+3}(F)/\gamma_{c+3}(F) \subseteq \mathcal{I}_{c+2}(N) = J^{c+2}$ and $\mathcal{B}_{c+2,J}$ is a $\mathbb{Z}$-basis of J^{c+2}, we have $\mathcal{B}_{c+2,J,f}\gamma_{c+2}(F)/\gamma_{c+2}(F)$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{I}_{c+2}(N)$. Thus

$$(b_1^{(c+2)})^{n_1} \cdots (b_{m(c+2)}^{(c+2)})^{n_{m(c+2)}} \notin \gamma_{c+3}(F) \tag{36}$$

for any $n_1, \ldots, n_{m(c+2)} \in \mathbb{Z}$ with $n_1 + \cdots + n_{m(c+2)} \neq 0$. The set $\mathcal{B}_{c+2,J,f}\gamma_{c+3}(F)/\gamma_{c+3}(F)$ is a $\mathbb{Z}$-basis of $\mathcal{I}_{c+2}(N)$. By (36), $\mathcal{B}_{c+2,f} \cap \gamma_{c+3}(F) = \emptyset$. Write $N_{(c+2)1}$ and $N_{(c+2)2}$ for the subgroups of N_{c+2} generated by the sets $\mathcal{B}_{c+2,J,f}$ and $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}_{\mathcal{V}}, s \geq c+1, g_1, \ldots, g_s \in F \setminus \{1\}\}$, respectively. Note that $N_{(c+2)2}$ is normal in N_{c+2}. We claim that $N_{c+2} = N_{(c+2)1}N_{(c+2)2}$. It is enough to show that $N_{c+2} \subseteq N_{(c+2)1}N_{(c+2)2}$. Furthermore, it is enough to show that $(r^{\pm 1}, a_{i_1}, \ldots, a_{i_c}) \in N_{(c+2)1}N_{(c+2)2}$ for $r \in \mathcal{R}_{\mathcal{V}}$, $i = 1, \ldots, 6$. Since

$$(a^{-1}, b, c) = ((a,b)^{-1}, c)(((a,b)^{-1}, c), ((a,b)^{-1}, a^{-1}))(((a,b)^{-1}, a^{-1}), c)$$

and N_{c+2} is normal in F, it is enough to show that $(r, a_{i_1}, \ldots, a_{i_c}) \in N_{(c+2)1}N_{(c+2)2}$ for $r \in \mathcal{R}_{\mathcal{V}}$, $i = 1, \ldots, 6$. For $i = 1, \ldots, 6$ and $r \in \mathcal{R}_{\mathcal{V}}$, there are unique $n_{1i_1\cdots i_c}, \ldots, n_{m(c+2)i_1\cdots i_c} \in \mathbb{Z}$ such that

$$(r, a_{i_1}, \ldots, a_{i_c})\gamma_{c+3}(F) = (b_1^{(c+2)})^{n_{1i_1\cdots i_c}} \cdots (b_{m(c+2)}^{(c+2)})^{n_{m(c+2)i_1\cdots i_c}}\gamma_{c+3}(F).$$

As in the case $c = 2$, working the elements $b_1^{(c+2)}, \ldots, b_{m(c+2)}^{(c+2)}$ in $N_{(c+2)1}N_{(c+2)2} \subseteq \gamma_{c+2}(F)$ (Since the elements $b_1^{(c+2)}, \ldots, b_{m(c+2)}^{(c+2)}$ are regarded in N_{c+2}.) and using the identity $ab = ba(a,b)$, we form the product $(b_1^{(c+2)})^{n_{1i_1\cdots i_c}} \cdots (b_{m(c+2)}^{(c+2)})^{n_{m(c+2)i_1\cdots i_c}}$ in such a way

$$(r, a_{i_1}, \ldots, a_{i_c}) = (b_1^{(c+2)})^{n_{1i_1\cdots i_c}} \cdots (b_{m(c+2)}^{(c+2)})^{n_{m(c+2)i_1\cdots i_c}}v$$

with $v \in \gamma_{c+3}(F)$. Following the procedure, it is clearly enough that $v \in N_{(c+2)1}N_{(c+2)2}$. Thus $(r, a_{i_1}, \ldots, a_{i_c}) \in N_{(c+2)1}N_{(c+2)2}$ for all $r \in \mathcal{R}_{\mathcal{V}}$ and $i \in \{1, \ldots, 6\}$. Therefore $N_{c+2} = N_{(c+2)1}N_{(c+2)2}$. Since $N_{c+2} = N_{(c+2)1}N_{(c+2)2}$ and $N_{(c+2)2} \subseteq \gamma_{c+3}(F)$, we have, by the modular law,

$$N_{c+3} = N_{c+2} \cap \gamma_{c+3}(F) = (N_{(c+2)1}N_{(c+2)2}) \cap \gamma_{c+3}(F) = (N_{(c+2)1} \cap \gamma_{c+3}(F))N_{(c+2)2}.$$

By (36), we get $N_{(c+2)1} \cap \gamma_{c+3}(F)$ is generated by elements of length at least $2(c+1)$. Having in mind the way in which $b_1^{(c+2)}, \ldots, b_{m(c+2)}^{(c+2)}$ are regarded elements in N_{c+3} and the definition of $N_{(c+2)2}$, we have $N_{(c+2)1} \cap \gamma_{c+3}(F) \subseteq N_{(c+2)2}$. Thus N_{c+3} is generated by the set $\{(r^{\pm 1}, g_1, \ldots, g_s) : r \in \mathcal{R}, s \geq c+1, g_1, \ldots, g_{c+1} \in F \setminus \{1\}\}$ and so, $\mathcal{I}_{c+3}(N) = J^{c+3}$. $\square$

Since F is residually nilpotent, we have $\bigcap_{d \geq 2} N_d = \{1\}$. Since $N \subseteq F'$, we get $\mathcal{I}_1(N) = 0$. Since $\mathcal{I}_d(N) \cong N_d/N_{d+1}$ as $\mathbb{Z}$-modules for all $d \geq 2$, we have from Proposition 2, $N_d \neq N_{d+1}$ for all $d \geq 2$. Define

$$\mathcal{I}(N) = \bigoplus_{d \geq 2} N_d\gamma_{d+1}(F)/\gamma_{d+1}(F) = \bigoplus_{d \geq 2} \mathcal{I}_d(N).$$

Since N is a normal subgroup of F, we have $\mathcal{I}(N)$ is an ideal of L (see [13]).

Corollary 5 *$\mathcal{I}(N) = J$.*

Proof. Since $J = \bigoplus_{d \geq 2} J^d$ and $\mathcal{I}_2(N) = J^2$, we have from Proposition 2 that $\mathcal{I}(N) = J$. $\square$

Our next result gives a presentation of $\L(M_3)$.

Theorem 3 *$L/J \cong \L(M_3)$ as Lie algebras. Consequently, $\L(M_3)$ is a torsion-free Lie algebra.*

Proof. Recall that

$$Ł(M_3) = \bigoplus_{c\geq 1} \gamma_c(M_3)/\gamma_{c+1}(M_3).$$

Since $M_3/M_3' \cong F/NF' = F/F'$, we have $Ł(M_3)$ is generated as a Lie algebra by the set $\{\alpha_i : i = 1, \ldots, 6\}$ with $\alpha_i = a_i M_3'$. Since L is a free Lie algebra of rank 6 with a free generating set $\{x_1, \ldots, x_6\}$, the map ψ from L into $Ł(M_3)$ satisfying the conditions $\psi(x_i) = \alpha_i$, $i = 1, \ldots, 6$, extends uniquely to a Lie algebra homomorphism. Since $Ł(M_3)$ is generated as a Lie algebra by the set $\{\alpha_i : i = 1, \ldots, 6\}$, we have ψ is onto. Hence $L/\mathrm{Ker}\psi \cong Ł(M_3)$ as Lie algebras. By definition, $J \subseteq \mathrm{Ker}\psi$, and so ψ induces a Lie algebra epimorphism $\overline{\psi}$ from L/J onto $Ł(M_3)$. In particular, $\overline{\psi}(x_i + J) = \alpha_i$, $i = 1, \ldots, 6$. Note that $\overline{\psi}$ induces $\overline{\psi}_c$, say, a $\mathbb{Z}$-linear mapping from $(L^c + J)/J$ onto $\gamma_c(M_3)/\gamma_{c+1}(M_3)$. For $c \geq 2$,

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong \gamma_c(F)\gamma_{c+1}(F)N/\gamma_{c+1}(F)N \cong \gamma_c(F)/(\gamma_c(F) \cap \gamma_{c+1}(F)N).$$

Since $\gamma_{c+1}(F) \subseteq \gamma_c(F)$, we have by the modular law,

$$\gamma_c(F)/(\gamma_c(F) \cap \gamma_{c+1}(F)N) = \gamma_c(F)/\gamma_{c+1}(F)N_c.$$

But, by Proposition 2,

$$\gamma_c(F)/\gamma_{c+1}(F)N_c \cong (\gamma_c(F)/\gamma_{c+1}(F))/\mathcal{I}_c(N) \cong L^c/J^c.$$

Therefore

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong L^c/J^c \cong (L^c)^*.$$

Hence $\mathrm{rank}(\gamma_c(M_3)/\gamma_{c+1}(M_3)) = \mathrm{rank}(L^c)^*$. Since $J = \bigoplus_{m\geq 2} J^m$, we have $(L^c + J)/J \cong L^c/(L^c \cap J) = L^c/J^c \cong (L^c)^*$ (by Theorem 2), we obtain $\mathrm{Ker}\overline{\psi}_c$ is torsion-free. Since $\mathrm{rank}(\gamma_c(M_3)/\gamma_{c+1}(M_3)) = \mathrm{rank}(L^c)^*$, we have $\mathrm{Ker}\overline{\psi}_c = \{1\}$ and so, $\overline{\psi}_c$ is isomorphism. Since $\overline{\psi}$ is epimorphism and each $\overline{\psi}_c$ is isomorphism, we have $\overline{\psi}$ is isomorphism. Hence $L/J \cong Ł(M_3)$ as Lie algebras. □

Corollary 6 *$Ł(M_3)$ admits the presentation in [9] described by the original presentation of M_3.*

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